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CHANGING COFINALITIES AND THE NONSTATIONARY IDEAL

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ABSTRACT

A κ-c.c. iteration of a Prikry type forcing notion is defined. Applications to the nonstationary ideal are given.

§0. Introduction

We define a κ-c.c. iteration of a Prikry type forcing notion. Such iteration looks much easier when applied to problems connected with extending elementary embeddings, than the Magidor iteration of the Prikry forcing [Ma 1]. Using this iteration, we define a natural forcing for changing cofinality of a cardinal without adding new bounded sets, which replace Mitchell's complete iteration and decoupling [Mi 2].

As an application, we prove the following two theorems.

THEOREM I. *Assume G.C.H. Let μ a regular cardinal and for some $\kappa \geqq \mu$ there exists a sequence of normal ultrafilters on κ*

$$U_0 \lhd U_1 \lhd \cdots \lhd U_\alpha \lhd \cdots \qquad (\alpha < \mu')$$

where $\mu' = \mu + 1$ if $\mu < \kappa$ and $\mu' = \kappa$ otherwise,

so that U_0 concentrates on the set $\{\alpha < \kappa \mid \alpha$ is α^{++}-strongly compact (or α^+-supercompact)$\}$.

Then there exists a forcing notion P so that in V^P the following holds:

(a) $\mu^+ = \kappa$ *if* $\mu < \kappa$;

(b) *all the cardinals below $(\mu^+)^V$ and above κ are preserved if $\mu < \kappa$, and κ remains inaccessible if $\kappa = \mu$;*

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(c) *G.C.H*;
(d) NS_κ (*the nonstationary ideal on* κ) *is precipitous.*

Models with NS_κ precipitous for $\kappa = \aleph_1, \aleph_2$ were previously known, see [J–Ma–Mi–P], [G 2]. Also H. Woodin constructed a model with precipitous ideal over $\mathcal{P}_\kappa(\kappa^+)$ whose projection to κ is NS_κ, for an inaccessible κ. By T. Jech [J 2], for $\mu > \aleph_2$, if it is possible to drop the assumption that U_0 concentrates on the set of $\{\alpha < \kappa \mid \alpha$ is α^{++}-strongly compact or α is α^+-supercompact$\}$ in the statement of Theorem I, then the equiconsistency result holds.

Shortly after Theorem I was proved, M. Foreman, M. Magidor and S. Shelah [F–M–S] found another way of making NS_κ precipitous. They proved that after collapsing a supercompact (or even a superstrong) cardinal to κ^+ the NS_κ became precipitous. It is unclear which initial assumption is weaker. Our feeling is that the method of Theorem I should give the equiconsistency result.

THEOREM II. *Assume G.C.H. Let* κ *be a cardinal so that there exists a sequence of normal ultrafilters on* κ

$$U_0 \lhd U_1 \lhd \cdots \lhd U_\alpha \lhd \cdots \qquad (\alpha < \kappa)$$

s.t. U_0 *concentrates on the set* $\{\alpha < \kappa \mid \alpha$ *is* α^+*-supercompact*$\}$.

Then there exists a forcing notion P so that in V^P *the following holds*:
(a) κ *remains inaccessible*;
(b) *G.C.H.*;
(c) *there exists a stationary subset S of* κ *so that*
 (1) *for every regular cardinal* $\alpha < \kappa$, $S \cap \{\beta < \kappa \mid \mathrm{cof}\,\beta = \alpha\}$ *is stationary,*
 (2) $\mathrm{NS}_\kappa \restriction S$ (*the nonstationary ideal restricted to S, i.e. the set of all* $A \subseteq \kappa$ *s.t.* $A \cap S$ *is nonstationary*) *is* κ^+*-saturated.*

The following easily follows from Theorem II.

COROLLARY. *In* V^P, *the forcing with* $\mathrm{NS}_\kappa \restriction S$ *preserves all the cardinals, it does not add new bounded subsets to* κ, *and for every regular* $\alpha < \kappa$ *there exists a condition forcing* "$\mathrm{cf}\,\kappa = \alpha$".

Previously, H. Woodin starting from a measurable κ constructed a model with $\mathrm{NS}_\kappa \restriction S$ κ^+-saturated for $S \subseteq \{\alpha \mid \alpha$ is regular$\}$. By [Mi 2] we need at least a measurable κ of the Mitchell order κ for the conclusion of Theorem II. Once more we conjecture that the assumption, that U_0 concentrates on the set $\{\alpha < \kappa \mid \alpha$ is α^+-supercompact$\}$, can be dropped in Theorem II. And so the equiconsistency holds. It is unknown if the full nonstationary ideal can be saturated over an inaccessible.

The paper is organized as follows. In §1 we define a κ-c.c. iteration of forcing notions satisfying the Prikry condition. In §2 the forcing for adding a sequence of order type $\omega \cdot \omega$ is defined. It already contains the ideas needed for adding sequences of higher order type, which is done in §3. In §4 and §5 models for Theorems I and II are constructed.

Our notations are quite standard. We refer to T. Jech's book [J 1] and the paper of A. Kanamori and M. Magidor [K–Ma] for basic definitions and notions.

§1. κ-c.c. iteration of forcing notions satisfying the Prikry condition

Let $\langle P, \leqq \rangle$ be a forcing notion whose conditions are pairs of the form $\langle p, A \rangle$, where p is a finite sequence. Let us call a condition $\langle q, B \rangle$ a Prikry extension of $\langle p, A \rangle$ (or just P-extension) if $\langle q, B \rangle \geqq \langle p, A \rangle$ (is stronger) and $q = p$. We say that $\langle P, \leqq \rangle$ has the Prikry property (or satisfies the Prikry condition) if for any $\langle p, A \rangle \in P$ and a statement σ of the forcing language there exists a Prikry extension $\langle p, A' \rangle$ of $\langle p, A \rangle$ deciding σ, i.e. $\langle p, A' \rangle \Vdash \sigma$ or $\langle p, A' \rangle \Vdash \neg \sigma$. Notice that any forcing is isomorphic to the forcing of the above type. Just add the empty sequence to its conditions.

We shall shrink the class of forcing notions we are interested in as follows. For $\langle P, \leqq \rangle$ as above and a cardinal α, let us say that $\langle P, \leqq \rangle$ is α-weakly closed if for every increasing sequence $\langle \langle p, B_\beta \rangle \mid \beta < \gamma < \alpha \rangle$ of elements of P there exists $\langle p, B \rangle \in P$ stronger than every $\langle p, B_\beta \rangle$, $\beta < \gamma$.

Clearly, every α-closed forcing satisfies the Prikry condition and it is α-weakly closed. But also the Prikry forcing, the supercompact and strongly compact Prikry forcings on α are such. For appropriate α, the Magidor and the Radin forcings, and the Magidor iteration of the Prikry forcing have these properties; see [Ma 2], [R], [Ma 1]. In §2, §3 we shall define other forcing notions of this type. By K. Prikry [P], an α-weakly closed forcing having the Prikry property does not add new bounded subsets to α.

We are now going to define the iteration of forcing notions of such a kind.

Let A be a set consisting of inaccessible cardinals. Denote by A' the closure of the set $A \cup \{\alpha + 1 \mid \alpha \in A\}$. For every $\alpha \in A'$ define by induction $\mathscr{P}_\alpha$ to be the set of all elements p of the form $\{\langle g_\gamma, A_\gamma \rangle \mid \gamma \in g\}$, where

(1) g is a subset of $\alpha \cap A$;

(2) g has an Easton support, i.e. for every inaccessible $\beta \leqq \alpha$, $\beta > |\mathrm{dom}\, g \cap \beta|$;

(3) for every $\gamma \in \mathrm{dom}\, g$

$$p \restriction \gamma = \{\langle g_\beta, A_\beta \rangle \mid \beta \in \gamma \cap g\} \in \mathscr{P}_\gamma$$

and

> $p \restriction \gamma \Vdash_{P_\gamma}$ "$\mathbf{g}_\gamma$ is a finite sequence, $\langle \mathbf{g}_\gamma, \mathbf{A}_\gamma \rangle$ is a condition in a $\check{\gamma}$-weakly closed forcing notion $\mathbf{Q}_\gamma$ satisfying the Prikry condition and if $\check{\gamma}'$ is the least element of $\check{A}$ above $\check{\gamma}$, then $|\mathbf{Q}_\gamma| \leqq \check{\gamma}'$".

Let $p = \langle \mathbf{g}_\gamma, \mathbf{A}_\gamma \rangle \mid \gamma \in g\}$, $q = \{\langle \mathbf{f}_\gamma, \mathbf{B}_\gamma \rangle \mid \gamma \in f\}$ be elements of $\mathcal{P}_\alpha$. Then $p \geqq q$ (p is stronger than q) if the following holds:

(1) $g \supseteq f$;

(2) for every $\gamma \in f$

$$p \restriction \gamma \Vdash_{\mathcal{P}_\gamma} \text{“}\langle \mathbf{f}_\gamma, \mathbf{B}_\gamma \rangle \leqq \langle \mathbf{g}_\gamma, \mathbf{A}_\gamma \rangle \text{ in the forcing } \mathbf{Q}_\gamma\text{”};$$

(3) There exists a finite subset b of f so that for every $\gamma \in f - b$, $p \restriction \gamma \Vdash_{\mathcal{P}_\gamma}$ "$\mathbf{f}_\gamma = \mathbf{g}_\gamma$".

REMARK. The main difference between $\mathcal{P}_\alpha$ and the Magidor iteration of the Prikry forcing [Ma 1] is that in $\mathcal{P}_\alpha$, instead of the full support in the second coordinate, only the Easton support is allowed. We shall show that $\mathcal{P}_\alpha$ still has most of the nice properties of the Magidor iteration and in addition it satisfies α-c.c.

Let $\alpha \in A'$, $p = \{\langle \mathbf{g}_\gamma, \mathbf{A}_\gamma \rangle \mid \gamma \in g\} \in \mathcal{P}_\alpha$. For $\beta \in \alpha \cap A'$, $p \restriction \beta = \{\langle \mathbf{g}_\gamma, \mathbf{A}_\gamma \rangle \mid \gamma \in \beta \cap g\}$ is a condition in $\mathcal{P}_\beta$. In the same fashion let us denote for every $\beta < \alpha$ by $p \restriction \beta$ the set $\{\langle \mathbf{g}_\gamma, \mathbf{A}_\gamma \rangle \mid \gamma \in \beta \cap g\}$ and by $\mathcal{P}_\beta$ the set $\{p \restriction \beta \mid p \in \mathcal{P}_\alpha\}$. Denote also by $p \setminus \beta$ the set $\{\langle \mathbf{g}_\gamma, \mathbf{A}_\gamma \rangle \mid \gamma \in g \setminus \beta\}$.

DEFINITION 1.1. Let $\alpha \in A'$ and $p = \{\langle \mathbf{g}_\gamma, \mathbf{A}_\gamma \rangle \mid \gamma \in g\}$, $q = \{\langle \mathbf{f}_\gamma, \mathbf{B}_\gamma \rangle \mid \gamma \in f\}$ be elements of $\mathcal{P}_\alpha$. Then p is an Easton extension (or E-extension) of q ($p_E \geqq q$) if $p \geqq q$ and for every $\gamma \in f$, $p \restriction \gamma \Vdash_{\mathcal{P}_\gamma}$ "$\mathbf{g}_\gamma = \mathbf{f}_\gamma$".

For an ordinal β, p is an Easton extension of q above β if $p \geqq q$ and for every $\gamma \in f - (\beta + 1)$, $p \restriction \gamma \Vdash_{\mathcal{P}_\gamma}$ "$\mathbf{g}_\gamma = \mathbf{f}_\gamma$".

LEMMA 1.2. *Let $\langle p_\sigma \mid \sigma < \beta \rangle$ be a sequence of elements of $\mathcal{P}_\alpha$ so that $p_{\sigma_1} \restriction (\beta + 1) = p_{\sigma_2} \restriction (\beta + 1)$ and $p_{\sigma_1\, E} \geqq p_{\sigma_2}$ for every $\sigma_1 \geqq \sigma_2$. Then there exists $p \in \mathcal{P}_\alpha$ so that $p \restriction (\beta + 1) = p_\sigma \restriction (\beta + 1)$ and $p_E \geqq p_\sigma$ for every $\sigma < \beta$.*

PROOF. Let $p_\sigma = \{\langle \mathbf{g}_{\gamma,\sigma}, \mathbf{A}_{\gamma,\sigma} \rangle \mid \gamma \in g_\sigma\}$ for $\sigma < \beta$. Define g to be $\bigcup\{g_\sigma \mid \sigma < \beta\}$. The equality $p_{\sigma_1} \restriction (\beta + 1) = p_{\sigma_2} \restriction (\beta + 1)$ implies that g has an Easton support. For $\gamma \in (\beta + 1) \cap g$ all $\mathbf{A}_{\gamma,\sigma}$, $\mathbf{g}_{\gamma,\sigma}$ are the same. Set $\mathbf{B}_\gamma = \mathbf{A}_{\gamma,0}$ and $\mathbf{f}_\gamma = \mathbf{g}_{\gamma,0}$, for such γ's. Define

$$p \restriction (\beta + 1) = \{\langle \mathbf{f}_\gamma, \mathbf{B}_\gamma \rangle \mid \gamma \in \operatorname{dom} g \cap (\beta + 1)\}.$$

Continue the definition of p by induction. Suppose that for some τ, $p \restriction \tau$ is defined so that $p \restriction \tau_E \geqq p_\sigma \restriction \tau$ $(\sigma < \beta)$. If there are now elements of g above τ, then we are finished. Otherwise let γ be $\min(g - \tau)$. Set $p \restriction \gamma = p \restriction \tau$. Let $\rho < \beta$ be the minimal s.t. $\gamma \in g_\rho$. Then

$$p \restriction \gamma \Vdash_{\mathcal{P}_\gamma} \text{“}\langle\langle \boldsymbol{g}_{\gamma,\rho}, \boldsymbol{A}_{\gamma,\sigma}\rangle \mid \rho \leqq \sigma \leqq \beta\rangle \text{ is an increasing sequence}$$

$$\text{of conditions in a } \gamma\text{-weakly closed forcing notion } \boldsymbol{Q}_\gamma\text{”}.$$

So for some $\mathcal{P}_\gamma$-name $\boldsymbol{B}_\gamma$, $p \restriction \gamma \Vdash_{\mathcal{P}_\gamma}$ “$\langle \boldsymbol{g}_{\gamma,\rho}, \boldsymbol{B}_\gamma\rangle$ is stronger than every $\langle \boldsymbol{g}_{\gamma,\rho}, \boldsymbol{A}_{\gamma,\sigma}\rangle$ $(\rho < \sigma < \beta)$”. Set $p \restriction (\gamma+1) = p \restriction \gamma \cup \{\langle \boldsymbol{g}_{\gamma,0}, \boldsymbol{B}_\gamma\rangle\}$. It completes the inductive definition. □

LEMMA 1.3. *Let α be a limit point of A. If α is a Mahlo cardinal, then $\mathcal{P}_\alpha$ satisfies α-c.c.*

The proof uses the standard Δ-system argument.

LEMMA 1.4. *Let $\alpha \in A'$, $p \in \mathcal{P}_\alpha$ and σ be a statement in the forcing language appropriate for $\mathcal{P}_\alpha$. Then there is an Easton extension p^* of p deciding σ, i.e. $p^* \Vdash \sigma$ or $p^* \Vdash \neg \sigma$.*

REMARK. The lemma is a weak analog of the Prikry condition for $\mathcal{P}_\alpha$.

PROOF. Let $p = \{\langle \boldsymbol{g}_\gamma, \boldsymbol{A}_\gamma\rangle \mid \gamma \in g\}$ and β be the minimal element in g. We assume that $g \neq \emptyset$, otherwise simply replace p by some of its E-extension with $g \neq \emptyset$.

Let G be a generic subset of $\mathcal{P}_{\beta+1}$, so that $p \restriction \beta+1 \in G$. We shall mean by $p \setminus (\beta+1) = \{\langle \boldsymbol{g}_\gamma, \boldsymbol{A}_\gamma\rangle \mid \gamma \in g - (\beta+1)\}$ the interpretation of it in $V[G]$, i.e. an element in the forcing $\mathcal{P}_\alpha / G$. Define now $p^* \in \mathcal{P}_\alpha / G$. If there exists some $q \in \mathcal{P}_\alpha / G$, an E-extension of $p \setminus (\beta+1)$ deciding σ, then set p^* to be some such q. Otherwise set $p^* = p \setminus (\beta+1)$. Then p^* looks like $\{\langle \boldsymbol{g}^*_\gamma, \boldsymbol{A}^*_\gamma\rangle \mid \gamma \in g^*\}$, for some $g^* \in V$ and $\mathcal{P}_\alpha / G$-names $\boldsymbol{g}^*_\gamma$, $\boldsymbol{A}^*_\gamma$. Let $\boldsymbol{p}^*$ be a $\mathcal{P}_{\beta+1}$-name of p^*. We would like to turn $\boldsymbol{p}^*$ into a condition. The problem is that different elements of $\mathcal{P}_{\beta+1}$ can force g^* to be different sets in V. But we can just take the union of all such possible sets. By the cardinalities assumption on A and $\mathcal{P}_{\alpha+1}$, this union will still have an Easton support.

So let us assume from the beginning that the set g^* in $\boldsymbol{p}^*$ is not a name of a subset of A, but it is really a subset of A. Then $\boldsymbol{p}^*$ will look like $\{\langle \boldsymbol{g}^*_\gamma, \boldsymbol{A}^*_\gamma\rangle \mid \gamma \in g^*\}$ for some $\mathcal{P}_\alpha$-names $\boldsymbol{g}^*_\gamma$ and $\boldsymbol{A}^*_\gamma$. Also $p \restriction (\beta+1) \cup \boldsymbol{p}^*$ will be a condition in $\mathcal{P}_\alpha$. Clearly, it is an E-extension of p.

Let φ be the statement “$\boldsymbol{p}^* \parallel \sigma$”. Since $\emptyset = p \restriction \beta \Vdash$ “Q_β satisfies the Prikry condition", for some $\mathscr{P}_\beta$-name $\boldsymbol{A}^*_\beta$

$$\varnothing \Vdash_{\mathscr{P}_\beta} \text{“}\langle \boldsymbol{g}_\beta, \boldsymbol{A}^*_\beta\rangle \parallel \varphi \text{ and } \langle \boldsymbol{g}_\beta, \boldsymbol{A}^*_\beta\rangle \geqq \langle \boldsymbol{g}_\beta, \boldsymbol{A}_\beta\rangle\text{”}.$$

Define now a $\mathscr{P}_\beta$-name $\boldsymbol{A}^{**}_\beta$.

Let G be a generic subset of $\mathscr{P}_\beta$. If, in $V[G]$, $\langle g_\beta, A^*_\beta\rangle \Vdash \neg\varphi$ then set $A^{**}_\beta = A^*_\beta$. Suppose now that $\langle g_\beta, A^*_\beta\rangle \Vdash \varphi$. Then let A^{**}_β be so that $\langle g_\beta, A^{**}_\beta\rangle \geqq \langle g_\beta, A^*_\beta\rangle$ and $\langle g_\beta, A^{**}_\beta\rangle$ decides "$p^* \Vdash \sigma$". Clearly, then $\langle g_\beta, A^{**}_\beta\rangle \Vdash \neg(p^* \Vdash \sigma)$ will imply that $\langle g_\beta, A^{**}_\beta\rangle \Vdash (p^* \Vdash \neg\sigma)$.

Now, return to V and define

$$p(1) = \{\langle g_\beta, \boldsymbol{A}^{**}_\beta\rangle\} \cup \boldsymbol{p}^*.$$

CLAIM 1.4.1. *Let q be an E-extension of $p(1)$ above β, then $q \Vdash {}^i\sigma$ iff*

$$q \restriction \beta \cup p(1) \setminus \beta \Vdash {}^i\sigma$$

where $i \in 2$ and ${}^0\sigma \equiv \sigma$, ${}^1\sigma \equiv \neg\sigma$.

PROOF. Let q be an E-extension of $p(1)$ above β and $q \Vdash \sigma$. The case $q \Vdash \neg\sigma$ is the same.

$q \restriction (\beta+1) \Vdash_{\mathscr{P}_{\beta+1}} (q \setminus (\beta+1)$ is an E-extension of $p \setminus (\beta+1)$ and it forces $\sigma)$. Then by the choice of $\boldsymbol{p}^*$

$$q \restriction (\beta+1) \Vdash (\boldsymbol{p}^* \text{ forces } \sigma).$$

So $q \restriction \beta \Vdash_{\mathscr{P}_\beta} (\langle \boldsymbol{q}_\beta, \boldsymbol{B}_\beta\rangle \Vdash_{\boldsymbol{Q}_\beta} (\boldsymbol{p}^* \text{ forces } \sigma))$ where $\langle \boldsymbol{q}_\beta, \boldsymbol{B}_\beta\rangle$ is the pair in q standing on the place β. By the choice of $\langle \boldsymbol{g}_\beta, \boldsymbol{A}^{**}_\beta\rangle$, then

$$q \restriction \beta \Vdash_{\mathscr{P}_\beta} (\langle \boldsymbol{g}_\beta, \boldsymbol{A}^{**}_\beta\rangle \Vdash_{\boldsymbol{Q}_\beta} (p^* \text{ forces } \sigma)).$$

Hence

$$q \restriction \beta \cup p(1) \setminus \beta \Vdash \sigma.$$ □ of the claim.

Let us define now an increasing continuous sequence of ordinals $\langle \beta(\sigma) \mid 0 < \sigma \leqq \eta\rangle$ and an E-increasing sequence $\langle p(\sigma) \mid \sigma \leqq \eta\rangle$ of elements of $\mathscr{P}_\alpha$ so that

(1) for every $q \geqq p(\sigma+1)$ which is an E-extension of $p(\sigma+1)$ above $\beta(\sigma+1)$

$$q \Vdash {}^i\sigma \text{ iff}$$

$$q \restriction \beta(\delta+1) \cup p(\delta+1) \setminus \beta(\delta+1) \Vdash {}^i\sigma$$

where $i \in 2$ and ${}^0\sigma \equiv \sigma$, ${}^1\sigma \equiv \neg\sigma$;

(2) $p(\delta+1) \restriction \beta(\delta+1) = p(\delta) \restriction \beta(\delta+1)$;

(3) $p(\delta)=\{\langle \mathbf{g}(\delta), \mathbf{A}_\gamma(\delta)\rangle \mid$ for some $g(\delta)\subseteq A$ and $\mathcal{P}_\alpha$-names $\mathbf{g}_\gamma(\delta)$, $\mathbf{A}_\gamma(\delta)$, $\gamma\in g(\delta)\}$;

(4) for a limit δ

$$g(\delta)=U\{g(\delta')\mid \delta'<\delta\};$$

(5) η is the least ordinal δ so that $g(\delta)=\{\beta(\delta')\mid \delta\geqq\delta'$, δ' is a successor ordinal$\}$;

(6) if $1<\delta+1\leqq\eta$, then

$$\beta(\delta+1)=\min(g(\delta)-(\beta(\delta)+1)).$$

Set $p(0)=p$ and $\beta(1)=\beta$. $p(1)$ is already defined and by Claim 1.4.1 it satisfies (1).

Suppose now that the sequences $\langle\beta(\delta')\mid 0<\delta'<\delta\rangle$ and $\langle p(\delta')\mid \delta'<\delta\rangle$ are defined. Assume that $\delta\leqq\eta$, i.e. there is no $\xi<\delta$ satisfying (5). Otherwise we are done.

Claim 1.4.2. *For every limit $\xi<\delta$, $\beta(\xi)$ is singular.*

Proof. Suppose otherwise. Then $\beta(\xi)=\xi$, since the sequence $\langle\beta(\delta')\mid\delta'<\xi\rangle$ is increasing and continuous. But $g(\xi)=U\{g(\delta')\mid\delta'<\xi\}$ and by (5), (6) $g(\xi)$ contains the set $\{\beta(\delta'+1)\mid\delta'<\xi\}$ of cardinality ξ. It is impossible, since $p(\xi)\in\mathcal{P}_\alpha$ and hence for the regular cardinal ξ, $|g(\xi)\cap\xi|<\xi$. □ of the claim.

Claim 1.4.3. *For every $\xi<\delta$, $g(\xi)-(\beta(\xi)+1)\neq\varnothing$.*

Proof. Since $\xi<\delta\leqq\eta$, by (5), (6)

$$g(\xi)\supsetneqq\{\beta(\delta'+1)\mid\delta'<\xi\}.$$

Pick some $\gamma\in g(\xi)-\{\beta(\delta'+1)\mid\delta'<\xi\}$.

Let us show that $\gamma>\beta(\xi)$. Suppose otherwise. Then $\gamma<\beta(\xi)$, by its definition if ξ is a successor ordinal, or by Claim 1.4.2 if ξ is a limit one. Pick the minimal $\xi'<\xi$ so that $\beta(\xi')\leqq\gamma<\beta(\xi'+1)$. As above $\gamma\neq\beta(\xi')$. By (6), $\beta(\xi'+1)=\min(g(\xi')-(\beta(\xi')+1))$. Hence $\gamma\notin g(\xi')$. But (2) implies that

$$g(\xi')\cap\beta(\xi'+1)=g(\xi'+1)\cap\beta(\xi'+1)=\cdots=g(\xi)\cap\beta(\xi'+1).$$

Hence $\gamma\notin g(\xi)$. Contradiction. □ of the claim

Suppose first that $\delta=\xi+1$. Then $g(\xi)-(\beta(\xi)+1)\neq\varnothing$, by Claim 1.4.3, and we define $\beta(\delta)$ as in (6). Define now $p(\delta)$ in the same fashion as $p(1)$ was defined.

Suppose that δ is a limit ordinal. Define $g(\delta)=\bigcup\{g(\delta')\mid\delta'<\delta\}$.

CLAIM 1.4.4. *For every inaccessible cardinal* λ, $|g(\delta)\cap\lambda|<\lambda$ *and* $\{\beta(\delta')\mid\delta'\leqq\delta\}\cap\lambda$ *is bounded in* λ.

PROOF. Let us show first that $\{\beta(\delta')\mid\delta'\leqq\delta\}\cap\lambda$ cannot be unbounded in λ. Suppose otherwise. Then, since $g(\delta)\supset\{\beta(\delta'+1)\mid\delta'<\delta\}$, $|g(\delta)\cap\lambda|=\lambda$.

Let $c(\delta')=\bigcup(g(\delta')\cap\lambda)$ for $\delta'<\delta$. Then c is a nondecreasing continuous function whose range is unbounded in λ.

Let $\xi=\{\delta'<\delta\mid\beta(\delta')<\lambda\}$. Then by Claim 1.4.2, $\xi=\delta$. Hence $\langle\beta(\delta')\mid\delta'<\delta\rangle$ is an increasing continuous sequence unbounded in λ. So there exists a limit ordinal $\delta_0<\lambda$ such that $\delta_0=c(\delta_0)=\beta(\delta_0)$. Then $\beta(\delta_0+1)=\min(g(\delta_0)-(\delta_0+1))>\lambda$. Contradiction. Hence $\{\beta(\delta')\mid\delta'\leqq\delta\}\cap\lambda$ is bounded in λ.

Let $\delta_0=\{\delta'<\delta\mid\beta(\delta')<\lambda\}$. Clearly $\delta_0<\lambda$. If $\delta_0=\delta$, then $|g(\delta)\cap\lambda|<\lambda$, since $|g(\xi)\cap\lambda|<\lambda$ for every $\xi<\delta$ and $g(\delta)=\bigcup\{g(\xi)\mid\xi<\delta\}$. Suppose that $\delta_0<\delta$. Then

$$\beta(\delta_0+1)=\min(g(\delta_0)-(\beta(\delta_0)+1))>\lambda.$$

Now (2) implies that $g(\delta_0)\cap\beta(\delta_0+1)=g(\delta)\cap\beta(\delta_0+1)$. Hence $g(\delta)\cap\lambda\subseteq\beta(\delta_0)+1<\lambda$. □ of the claim

Now let us define $p(\delta)$. Let $q\in\mathcal{P}_{\beta(\delta)}$ be so that $q\restriction\beta(\delta'+1)=p(\delta')\restriction\beta(\delta'+1)$ for every $\delta'<\delta$. By (2) such q exists. Set $p'(\delta')=q\cap p(\delta')\setminus\beta(\delta)$. Then $\langle p'(\delta')\mid\delta'<\delta\rangle$ is an E-increasing sequence so that $p'(\delta')\restriction(\beta(\delta)+1)=q$ for every $\delta'<\delta$. Let $p\in\mathcal{P}_\alpha$ be as in Lemma 1.3. Then $p\restriction(\beta(\delta)+1)=q$ and $p\,{}_E{\geqq}\,p'(\delta')$ for every $\delta'<\delta$.

Set $p(\delta)=p$. It completes the construction of the sequences $\langle\beta(\delta)\mid0<\delta\leqq\eta\rangle$ and $\langle p(\delta)\mid\delta\leqq\eta\rangle$.

Set now $p^*=p_\eta$. Let $p^*=\{\langle \boldsymbol{g}^*_\gamma,\boldsymbol{A}^*_\gamma\rangle\mid\gamma\in g^*\}$. If there exists some E-extension of p^* deciding σ, then we are done. Suppose otherwise. Let $q\geqq p^*$ be a condition deciding σ. Let $q=\{\langle \boldsymbol{f}_\gamma,\boldsymbol{B}_\gamma\rangle\mid\gamma\in f\}$.

By the definition of the extension there exists a finite subset b of g^* so that for every $\gamma\in g^*-b$

$$q\restriction\gamma\Vdash_{\mathcal{P}_\gamma}\text{“}\boldsymbol{f}_\gamma=\boldsymbol{g}^*_\gamma\text{”}.$$

Denote by $\gamma(q)$ the maximal element of b.

Pick a condition $s\geqq p^*$ deciding σ with $\gamma(s)$ the minimal possible. Since $\gamma(s)\in g^*$, by (5), $\gamma(s)=\beta(\delta+1)$ for some $\delta<\eta$. clearly, s is an E-extension of p^* above $\gamma(s)$. So, by (1)

$$s_1=s\restriction\beta(\delta+1)\cup p^*\setminus\beta(\delta+1)\|\sigma.$$

But $\gamma(s_1)$ is clearly less than $\gamma(s)$. Contradiction. It completes the proof of the lemma.

Notice, that already $s \upharpoonright \beta(1) \cup p^* \| \sigma$. □

LEMMA 1.5. *Let α be a limit point of A. Suppose that $2^\alpha = \alpha^+$, α is a measurable cardinal and there exists a normal ultrafilter U on α so that $A \cap \alpha \notin U$. Then U extends in $V^{\mathcal{P}_\alpha}$ to a normal ultrafilter.*

PROOF. Let j: $V \to N \simeq V^\alpha / U$ be the canonical elementary embedding. Let $\langle \mathbf{A}_\gamma \mid \gamma < \alpha^+ \rangle$ be an enumeration of all canonical $\mathcal{P}_\alpha$-names of subsets of α.

Pick G a generic subset of $\mathcal{P}_\alpha$. By the definition of $\mathcal{P}_\alpha$, $j''(G) = G$. Define by induction an E-increasing sequence $\langle p_\gamma \mid \gamma < \alpha \rangle$ of elements of $\mathcal{P}_{j(\alpha)}/G$ so that for every $\gamma < \alpha^+$, $p_{\gamma+1} \| \check{\alpha} \in j(\mathbf{A}_\gamma)$. Every initial segment of the sequence will lie in $N[G]$. On successor stages let us apply Lemma 1.4 and on limit, Lemma 1.2. Note that $\alpha \notin j(A)$ and so every forcing notion in the iteration $\mathcal{P}_{j(\alpha)}/G$ is α^+-weakly closed.

Define now an ultrafilter U' in $V[G]$ as follows. For $A \subseteq \kappa$ set $A \in U'$ if for some $\gamma < \alpha^+$, some $\mathcal{P}_\alpha$-name $\mathbf{A}$ of A in $N[G]$

$$p_\gamma \Vdash_{j(\alpha)/G} \check{\alpha} \in j(\mathbf{A}).$$

Since $j''(G) = G$, the definition really does not depend on a particular name of A. The checking that U' is a normal ultrafilter extending U is standard. □

The next lemma follows from Lemmas 1.2, 1.3 and 1.4.

LEMMA 1.6. *Let a Mahlo cardinal α be a limit point of A. Then for every $\beta \geqq \alpha$, α remains a cardinal in $V^{\mathcal{P}_\beta}$.*

By stengthening assumptions on $\mathbf{Q}_\alpha$'s ($\alpha \in A$) it is possible to say much more about preserving cardinals. For example, if $\varnothing \Vdash_{\mathcal{P}_\alpha}$ "$|\mathbf{Q}_\alpha| < \check{\alpha}'$" for every $\alpha \in A$, where α' is the least element of A above α (if there is any), then all elements of A will remain cardinals.

Let us describe briefly another way of iterating weakly closed forcing notions satisfying the Prikry condition. We shall shrink the class of allowed Q_α's. But it really will cover most of the applications below.

Define $\mathcal{P}_\alpha$ to be all p of the form $\{\langle g_\gamma, \mathbf{A}_\gamma \rangle \mid \gamma \in \operatorname{dom} g\}$, where:

(1) g is a partial Easton support function on $\alpha \cap A$, i.e. for every inaccessible $\beta \leqq \alpha$, $\beta > |\operatorname{dom} g \cap \beta|$;

(2) for every $\gamma \in \operatorname{dom} g$, $g(\gamma) = g_\gamma$ is a finite sequence and $p \mid \gamma = \{\langle g_\beta, \mathbf{A}_\beta \rangle \mid \beta \in \gamma \cap \operatorname{dom} g\}$,

$\Vdash_{\mathscr{P}_\gamma}$ "$\langle g_\gamma, \boldsymbol{A}_\gamma\rangle$ is a condition in a γ-weakly closed forcing notion $\boldsymbol{Q}_\gamma$ of cardinality γ' satisfying the Prikry condition, where γ' is the least element of A above γ or if γ is maximal in A, then $\gamma' = \infty$".

Let $p = \{\langle g_\gamma, \boldsymbol{A}_\gamma\rangle \mid \gamma \in \operatorname{dom} g\}$, $q = \{\langle f_\gamma, \boldsymbol{B}_\gamma\rangle \mid \gamma \in \operatorname{dom} f\}$ be elements of $\mathscr{P}_\alpha$. Then $p \geqq q$ (p is stronger than q) if the following holds:

(1) $\operatorname{dom} g \supseteq \operatorname{dom} f$;

(2) the number of γ's in $\operatorname{dom} f$, so that $g_\gamma \neq f_\gamma$, is finite;

(3) for every $\gamma \in \operatorname{dom} f$

$$p \upharpoonright \gamma \Vdash_{\mathscr{P}_\gamma} \text{“}\langle \check{f}_\gamma, \boldsymbol{B}_\gamma\rangle \leqq \langle \check{g}_\gamma, \boldsymbol{A}_\gamma\rangle \text{ in the forcing } \boldsymbol{Q}_\gamma\text{”}.$$

Let $\beta < \alpha$. Denote

$$\mathscr{P}_{\alpha\beta} = \{p \in \mathscr{P}_\alpha \mid p \text{ is of the form } \{\langle g_\gamma, \boldsymbol{A}_\gamma\rangle \mid \gamma \in \operatorname{dom} g\}, \text{ where } \boldsymbol{A}_\gamma \text{ does not depend on } \mathscr{P}_\beta \text{ for every } \gamma \in \operatorname{dom} g \setminus \beta\}.$$

Suppose that for every β, $\mathscr{P}_{\alpha.\beta+1}$ is dense in $\mathscr{P}_\alpha$. Under this assumption, Lemmas 1.2–1.5 can be proved in the present situation.

Such defined iteration can be used in §§2–4 but it is not good for the construction of a model with $\mathrm{NS}_\kappa \upharpoonright S$-saturated.

§2. Changing cofinalities without adding new bounded subsets: adding a sequence of the order type $\omega \cdot \omega$

We are going to define a forcing notion replacing Mitchell's complete iteration and decoupling [Mi 2]. We present first the simple case, adding a sequence of order type $\omega \cdot \omega$. It will contain most of the ideas needed for the general case.

Assume G.C.H. Let $U(\kappa,0)$, $U(\kappa,1)$ be two normal ultrafilters on κ so that $U(\kappa,0) \lhd U(\kappa,1)$ (i.e. $U(\kappa,0)$ belongs to the ultrapower by $U(\kappa,1)$). Fix a sequence of normal ultrafilters $\langle U(\beta,0) \mid \beta \in A\rangle$ representing $U(\alpha,0)$ in the ultrapower, for some $A \subseteq \kappa$, $A \in U(\kappa,1) - U(\kappa,0)$. W.l.o.g. assume that for every $\beta \in A$, $A \cap \beta \notin U(\beta,0)$.

In the first stage we shall change the cofinality of every $\alpha \in A$ to ω iterating the Prikry forcing.

Set $\mathscr{P}_{\alpha_0+1}$ = the Prikry forcing with $U(\alpha_0,0)$ where α_0 is the least element of A. We refer to [P] or [K–Ma] or [J 1] for the definition and the properties of this forcing notion.

If α is a limit point of A, then let $\mathscr{P}_\alpha$ be the iteration defined in §1. Let α' be the least element of $A \geqq \alpha$ (if $\alpha \neq \kappa$). If $\alpha' > \alpha$ then by Levy–Solovay [L–S]

$$\bar{U}(\alpha',0) = \{X \subseteq \alpha' \mid X \in V^{\mathscr{P}_\alpha}, \text{ there exists } X' \in U(\alpha',0)\ X \supseteq X'\}$$

is a normal ultrafilter on α' in $V^{\mathcal{P}_\alpha}$. Set $Q_{\alpha'}$ to be the Prikry forcing with $\bar{U}(\alpha',0)$ and $\mathcal{P}_{\alpha'} = \mathcal{P}_\alpha * Q_{\alpha'}$.

Suppose now that $\alpha' = \alpha$, i.e. $\alpha \in A$ is a limit point of A. Then by Lemma 1.5 $U(\alpha,0)$ extends to a normal ultrafilter in $V^{\mathcal{P}_\alpha}$. For our purpose we need to pick an extension of $U(\alpha,0)$ more carefully. Let us fix from the beginning some wellordering W of V_λ (the sets in V of rank $<\lambda$) s.t. for every inaccessible $\beta<\lambda$

$$W \mid V_\beta \colon V_\beta \Leftrightarrow \beta,$$

where λ is a cardinal much above all the cardinals we are dealing with. Now, the only changes we need in Lemma 1.5 are the following: Pick the sequence of all canonical $\mathcal{P}_\alpha$-names of subsets of α $\langle \boldsymbol{A}_\gamma \mid \gamma < \alpha^+\rangle$ to be a $j_0(W)$-least such sequence in N_0 and for $\gamma < \alpha^+$ pick also $\boldsymbol{p}_\gamma$ to be $j_0(W)$-least, where $j_0\colon V \to N_0 \simeq V^\alpha / U(\alpha,0)$.

Let $\bar{U}(\alpha,0)$ be such a defined ultrafilter. Set Q_α to be the Prikry forcing with $\bar{U}(\alpha,0)$ and $\mathcal{P}_{\alpha+1} = \mathcal{P}_\alpha * Q_\alpha$.

The second stage is to define in $V^{\mathcal{P}_\kappa}$ a forcing notion adding to κ a cofinal sequence of order type $\omega \cdot \omega$ without adding new bounded subsets of κ. We shall first extend the ultrafilters $U(\kappa,0)$ and $U(\kappa,1)$ in $V^{\mathcal{P}_\kappa}$.

Let $j_i\colon V \to N_i \simeq V^\kappa / U(\kappa,i)$ be the canonical elementary embedding, for $i \in 2$.

Pick $G \subseteq \mathcal{P}_\kappa$ as generic subset. Let $\bar{U}(\kappa,0)$ be the normal ultrafilter extending $U(\kappa,0)$, defined as Lemma 1.5 using $j_0(W)$ in N_0. Let $\bar{U}'(\kappa,0)$ be the ultrafilter having the same definition as $\bar{U}(\kappa,0)$ but defined in $N_1[G]$. Note that $U(\alpha,0)\in N_1$.

Lemma 2.1. $\bar{U}(\alpha,0) = \bar{U}'(\alpha,0)$.

Proof. Let $j_{01}\colon N_0 \to N_{01} \simeq N_0^{j_0(\kappa)} / j_0(U(\kappa,1))$ and $j_{10}\colon N_1 \to N_{10} \simeq N_1/U(\kappa,1)$ be the canonical elementary embeddings. Then $N_{01} = N_{10} =_{\mathrm{df}} N$ and the following diagram is commutative.

$$\begin{array}{ccc} V & \xrightarrow{\;j_0\;} & N_0 \\ {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle j_{01}} \\ N_1 & \xrightarrow{\;j_{10}\;} & N \end{array}$$

The first ordinal moved by j_{01} is $j_0(\kappa)$ and $N_0 \cap {}^{j_0(\kappa)}N \subseteq N$. Hence

$$j_{10}(j_1(W)) \mid V^N_{j_0(\kappa)} = j_{01}(j_0(W)) \mid V^N_{j_0(\kappa)} = j_0(W) \mid V^{N_0}_{j_0(\kappa)}.$$

So the minimal sequence $\langle \mathbf{A}_\gamma \mid \gamma < \kappa^+ \rangle \in N_0$ is also such for N. Clearly, $j_0(\kappa) = j_{10}(\kappa)$. Also, $\mathcal{P}_{j_0(\kappa)}$ is the same forcing notion in N_0 and N_1 since $\mathcal{P}_{j_0(\kappa)} \subseteq V^{V_0}_{j_0(\kappa)} = V^N_{j_0(\kappa)}$. Hence $\bar{U}(\alpha,0)$, $\bar{U}'(\alpha,0)$ have the same elements. □

Let us define now the ultrafilter $U(\kappa,1,t)$ extending $U(\kappa,1)$, for every finite increasing sequence t of ordinals less than κ.

In N_1,

$$\mathcal{P}_{j_1(\kappa)} = \mathcal{P}_{\kappa+1} * \mathcal{P}_{j_1(\kappa)}/\mathcal{P}_{\kappa+1}$$

and $\mathcal{P}_{\kappa+1} = \mathcal{P}_\kappa * Q_\kappa$, where, by Lemma 2.1, Q_κ is the Prikry forcing with $\bar{U}(\kappa,0)$. Let $\langle \mathbf{A}_\gamma \mid \gamma < \kappa^+ \rangle$ be the $j_1(W)$-least enumeration of all canonical $\mathcal{P}_\alpha$-names of subsets of α. Using Lemmas 1.3, 1.4 define a sequence of $\mathcal{P}_{\kappa+1}$-name of elements of $\mathcal{P}_{j_1(\kappa)}/\mathcal{P}_{\kappa+1}$ $\langle \mathbf{p}_\gamma \mid \gamma < \kappa^+ \rangle$ so that

(1) for limit $\gamma < \kappa^+$

$$\| \mathbf{p}_\gamma \text{ is the } j_1(W)\text{-least E-extension of } \langle \mathbf{p}_{\gamma'} \mid \gamma' < \gamma \rangle \|^{\mathcal{P}_{\kappa+1}} = 1,$$

(2) for every $\gamma < \kappa^+$

$$\| \mathbf{p}_{\gamma+1} \text{ is the } j_1(W)\text{-least E-extension of } \mathbf{p}_\gamma \text{ deciding ``} \check{\kappa} \in j_1(\mathbf{A}_\gamma)\text{''} \|^{\mathcal{P}_{\kappa+1}} = 1.$$

Work in $V[G]$. Define $U(\kappa,1,t)$, for a finite sequence t of ordinals less than κ, as follows:

$$C \in U(\kappa,1,t) \quad \text{if } C \subseteq \kappa \text{ and for some } r \in G,\quad \gamma < \kappa^+,$$
$$B \in \bar{U}(\kappa,0), \quad \text{in } N_1,$$
$$r \cup \{\langle \check{t}, \mathbf{B} \rangle\} \cup \mathbf{p}_\gamma \Vdash \check{\kappa} \in j_1(\mathbf{C}),$$

for some names $\mathbf{C}$, $\mathbf{B}$ of C and B.

Lemma 2.2. *$U(\kappa,1,t)$ is a κ-complete ultrafilter extending $U(\kappa,1)$.*

Proof. The definition of $U(\kappa,1,t)$ does not depend on a particular name $\mathbf{C}$ for C, since by the definition of $\mathcal{P}_\kappa$ $j_1''(G) = G$. Also the definition is independent of the choice of B and its name $\mathbf{B}$, since conditions $\langle t, B_1 \rangle$, $\langle t, B_2 \rangle$ are always compatible, namely $\langle t, B_1 \cap B_2 \rangle$ is stronger than both of them.

Clearly, $U(\kappa,1,t) \supseteq U(\kappa,1)$.

Let us show that $U(\kappa,1,t)$ is a κ-complete ultrafilter. Let, for some $\alpha < \kappa$,

$$\varnothing \Vdash_{\mathcal{P}_\kappa} \text{``} \langle \mathbf{C}_\beta \mid \beta < \alpha \rangle \text{ is a sequence of subsets of } \kappa \text{ so that } \bigcup_{\beta<\alpha} \mathbf{C}_\beta = \kappa\text{''}.$$

Then, in N_1,

$$\varnothing \Vdash_{\mathcal{P}_{j_1(\kappa)}} \text{“}\langle j_1(\boldsymbol{C}_\beta) \mid \beta < \alpha\rangle \text{ is a sequence of subsets of } j_1(\kappa) \text{ so that } \bigcup_{\beta<\alpha} j_1(\boldsymbol{C}_\beta) = j_1(\kappa)\text{”}.$$

For every $\beta < \alpha$ there are $r \in G$ and $\alpha_\beta < \kappa^+$ so that $r \Vdash_{\mathcal{P}_\kappa} \boldsymbol{C}_\beta = \boldsymbol{A}_{\alpha_\beta}$. Let, in $V[G]$, $\gamma = \bigcup_{\beta<\alpha} \alpha_\beta$. Then for every $\beta < \alpha$

$$\| \boldsymbol{p}_\gamma \| \check{\kappa} \in j_1(\boldsymbol{C}_\beta) \|^{Q_\kappa} = 1.$$

Let $\boldsymbol{\gamma}$ be a $\mathcal{P}_\kappa$-name of such defined in $V[G]$ ordinal γ. Since $\mathcal{P}_\kappa$ satisfies κ-c.c. there exists an ordinal $\bar{\gamma} < \kappa^+$ so that $\varnothing \Vdash_{\mathcal{P}_\kappa} \check{\bar{\gamma}} \geqq \boldsymbol{\gamma}$. Hence

$$\| \boldsymbol{p}_{\bar{\gamma}} \| \check{\kappa} \in j_1(\boldsymbol{C}_\beta) \|^{\mathcal{P}_{\kappa+1}} = 1, \qquad \text{for every } \beta < \alpha.$$

Consider in $V[G]$ Q_κ-statements $\sigma_\beta \equiv \boldsymbol{p}_{\bar{\gamma}} \Vdash \check{\kappa} \in j_1(\boldsymbol{C}_\beta)$ for $\beta < \alpha$. Then there exists $B \in U(\alpha, 0)$ so that $\langle t, B\rangle$ decides σ_β for every $\beta < \alpha$. Then there is $\bar{\beta} < \alpha$ s.t.

$$\langle t, B\rangle \Vdash_{Q_\kappa} \text{“}\boldsymbol{p}_{\bar{\gamma}} \Vdash \check{\kappa} \in j_1(\boldsymbol{C}_{\bar{\beta}})\text{”}.$$

Hence for some $r \in G$

$$r \Vdash_{\mathcal{P}_\kappa} (\langle \check{t}, \boldsymbol{B}\rangle \Vdash_{Q_\kappa} (\boldsymbol{p}_{\bar{\gamma}} \Vdash \check{\kappa} \in j_1(\boldsymbol{C}_{\bar{\beta}}))).$$

But then $C_{\bar{\beta}} \in U(\kappa, 1, t)$. □

Note that $U(\alpha, 1, t)$ is not normal since the set $A \in U(\alpha, 1, t)$ and every $\beta \in A$ is of cofinality ω in $V[G]$. But it follows by Baumgartner [B] and Lemma 1.3 that $U(\kappa, 1, t)$ contains all closed unbounded subsets of κ.

It is also not hard to see that $\bigcap\{U(\kappa, 1, t) \mid t \in [\kappa]^{<\omega}\}$ is a normal κ^+-saturated filter over κ.

A κ-complete ultrafilter containing all closed unbounded subsets of κ and concentrating on singular cardinals, and a κ^+-saturated normal filter concentrating on singular cardinals were constructed previously by Mitchell [Mi 2].

We are starting now to describe the forcing notion for adding an unbounded sequence to κ of the order type $\omega \cdot \omega$. Work in $V[G]$. Denote by b_β, for $\beta \in A$, the generic sequence added to β by G, i.e. $\{\beta\} \cup \{t \mid \text{for some } r \in \mathcal{P}_\beta \cap G, \text{ some } \mathcal{P}_\beta\text{-name } \boldsymbol{B}\ r \cup \{\langle \check{t}, \boldsymbol{B}\rangle\} \in G\}$. For finite subset $\eta \subseteq A$, set $b_\eta = U\{b_\beta \mid \beta \in \eta\}$.

Let us call a finite increasing sequence of ordinals $\langle \delta_0, \dots, \delta_{n-1}\rangle$ 2-coherent if, for every $i < n$ so that $\delta_i \in A$, $\{\delta_j \mid i^* \leqq j < i\}$ is an initial segment of b_{δ_i} where $i^* \leqq i$ is the minimal so that for every $i^* \leqq j < i$, $\delta_j \notin A$.

DEFINITION 2.5. A 2-tree T is a tree consisting of 2-coherent sequences so that for every $\eta \in T$ the set of immediate successors of η in T, $\mathrm{Suc}_T(\eta)$, is a union of two disjoint sets $\mathrm{Suc}_{T,0}(\eta)$ and $\mathrm{Suc}_{T,1}(\eta)$ such that

$$\mathrm{Suc}_{T,0}(\eta) \in \bar{U}(\kappa, 0) \quad \text{and} \quad \mathrm{Suc}_{T,1}(\eta) \in U(\kappa, 1, \eta(1)),$$

where for a 2-coherent sequence $\eta = \langle \delta_0, \dots, \delta_{n-1} \rangle$ $\eta(1)$ is the empty sequence, unless $\delta_{n-1} \notin A$. If $\delta_{n-1} \notin A$, then let i^* be the minimal $i < n$ s.t. for every $j \geqq i$, $j < n$, $\delta_j \notin A$. Set $\eta(1) = \langle \delta_{i^*}, \dots, \delta_{n-1} \rangle$.

For an element η of a 2-tree T, denote by T_η the set of all $\nu \in T$, ν extends η in T.

Define now the forcing notion.

DEFINITION 2.4. Let $\mathscr{P}(\kappa, 2)$ be the set of all pairs $\langle t, T \rangle$, where t is a 2-coherent sequence and for some 2-tree T', s.t. $t \in T'$, $t = T'_t$.

For $\langle t_1, T_1 \rangle$, $\langle t_2, T_2 \rangle$ in $\mathscr{P}(\kappa, 2)$, let $\langle t_2, T_2 \rangle \geqq \langle t_1, T_1 \rangle$ if there exists $\eta \in T_1$ so that

(a) $b_\eta = b_{t_2}$,

(b) for every $\nu \in T_2$, $\eta^\frown \{\delta \in \nu \mid \delta > \max \eta\}$ is an element of $T_{1\eta}$.

This definition is motivated after the Prikry forcing (namely the strong compact version of it) [P] and the Magidor forcing [Ma 2].

Notice that the finite sequence t in a condition $\langle t, T \rangle$ provides really the information about the infinite sequence b_t. We shall show in the next section in a more general situation that $\mathscr{P}(\kappa, 2)$ satisfies the Prikry condition. The completeness of the ultrafilters $\bar{U}(\kappa, 0)$ and $U(\kappa, 1, t)$, $t \in [\kappa]^{<\omega}$, implies that the forcing $\mathscr{P}(\kappa, 2)$ is κ-weakly closed. So $\mathscr{P}(\kappa, 2)$ will not add new bounded subsets to κ.

§3. Changing cofinalities without adding new bounded subsets: the general case

Assume G.C.H. Let $\vec{U}$ be a coherent sequence of ultrafilters, i.e. a function with domain of the form

$$\{(\alpha, \beta) \mid \alpha < l^{\vec{U}} \text{ and } \beta < 0^{\vec{U}}(\alpha)\}$$

for an ordinal $l^{\vec{U}}$, the length of $\vec{U}$, and a function $0^{\vec{U}}(\alpha)$, the order of $\vec{U}$ at α. For each pair $(\alpha, \beta) \in \operatorname{dom} \vec{U}$,

(1) $U(\alpha, \beta)$ is a normal ultrafilter on α, and

(2) if j^α_β: $V \to N^\alpha_\beta \simeq V^\alpha / U(\alpha, \beta)$ is the canonical embedding then

$$j^\alpha_\beta(\vec{U}) \restriction \alpha + 1 = \vec{U} \restriction (\alpha, \beta),$$

where

$$\vec{U} \upharpoonright \alpha = \vec{U} \upharpoonright \{(\alpha', \beta') \mid \alpha' < \alpha \text{ and } \beta' < 0^{\vec{U}}(\alpha')\}$$

and

$$\vec{U} \upharpoonright (\alpha, \beta) = \vec{U} \upharpoonright \{(\alpha', \beta') \mid (\alpha' < \alpha \text{ and } \beta' < 0^{\vec{U}}(\alpha')) \text{ or } (\alpha' = \alpha \text{ and } \beta' < \beta)\}.$$

The notion of the coherent sequence of ultrafilters was introduced by Mitchell [Mi 1]. We refer to his papers [Mi 1], [Mi 3] for more details on this subject.

Denote by $\mathrm{dom}_1 \vec{U}$ the set of all α's such that for some β, $(\alpha, \beta) \in \mathrm{dom}\, \vec{U}$. Let us assume that $\alpha \in \mathrm{dom}_1 \vec{U}$ implies that $0^{\vec{U}}(\alpha) \geqq 1$. Assume also that for every $\alpha \in \mathrm{dom}_1 \vec{U}$, $0^{\vec{U}}(\alpha) \leqq \alpha$.

We are going to add a cofinal sequence to every α in $\mathrm{dom}_1 \vec{U}$. The order type of this sequence will depend on $0^{\vec{U}}(\alpha)$. By induction on $\alpha \in \mathrm{closure}(\mathrm{dom}_1 \vec{U} \cup \{\beta + 1 \mid \beta \in \mathrm{dom}_1 \vec{U}\})$, we shall define forcing notions $\mathcal{P}_\alpha$. $\mathcal{P}_\alpha$ will take care of adding cofinal sequences to every element of $\alpha \cap \mathrm{dom}_1 \vec{U}$. For α a limit point of $\mathrm{dom}_1 \vec{U}$, define $\mathcal{P}_\alpha$ to be the iteration described in Section 1. If $\alpha \in \mathrm{dom}_1 \vec{U}$ but it is not a limit point of $\mathrm{dom}_1 \vec{U}$, then $0^{\vec{U}}(\alpha) = 1$ and the cardinality of the forcing below α is less than α. So, by Levy–Solovay [L–S], $U(\alpha, 0)$ generates the normal ultrafilter in the extension. Let $\mathcal{P}_\beta$ be the forcing below α. Set $\boldsymbol{Q}_\alpha$ to be the Prikry forcing with $U(\alpha, 0)$ in $V^{\mathcal{P}_\beta}$ and $\mathcal{P}_{\alpha+1} = \mathcal{P}_\beta * \boldsymbol{Q}_\alpha$. Suppose now that $\alpha \in \mathrm{dom}_1 \vec{U}$ is a limit point of $\mathrm{dom}_1 \vec{U}$. Let us define the forcing notion Q_α we shall use on α. Let G be a generic subset of $\mathcal{P}_\alpha$.

For $\beta < \alpha$, $\beta \in \mathrm{dom}_1 \vec{U}$ denote by b_β the generic sequence added to β, i.e. $U\{t \mid \text{for some } r \in \mathcal{P}_\beta \cap G, \text{ some } \mathcal{P}_\beta \text{ name } \boldsymbol{T}\, r \cup \{\langle \check{t}, \boldsymbol{T} \rangle\} \in G\} \cup \{\beta\}$. For a finite sequence $\eta \subset \mathrm{dom}_1 \vec{U} \cap \alpha$ set $b_\eta = \bigcup \{b_\beta \mid \beta \in \eta\}$. For $\beta \notin \mathrm{dom}_1 U$ set $b_\beta = \{\beta\}$.

Definition 3.1. Let $\beta \in \mathrm{dom}_1 \vec{U} \cap (\alpha + 1)$, $\gamma \leqq 0^U(\beta)$.

A finite increasing sequence $\langle \delta_0, \ldots, \delta_{n-1} \rangle$ of ordinals less than β is γ-coherent if

(1) $\gamma = 0$ implies that $\langle \delta_0, \ldots, \delta_{n-1} \rangle$ is the empty sequence,

(2) for every $i < n$, $0^{\vec{U}}(\delta_i) < \gamma$,

(3) for $i < n$, let $i^* \leqq i$ be the minimal s.t. for every j, $i^* \leqq j < i$, $0^{\vec{U}}(\delta_j) < 0^{\vec{U}}(\delta_i)$.

Then for every $i < n$, $U\{b_{\delta_j} \mid i^* \leqq j < i\}$ is an initial segment of b_{δ_i}.

The following lemma follows easily from the definition.

Lemma 3.2. *Let β, γ be as in 3.1 and $\langle \delta_0, \ldots, \delta_{n-1} \rangle$ be a γ-coherent sequence. Then*

(a) *$\langle \delta_0, \ldots, \delta_i \rangle$ is a γ-coherent sequence for every $i < n$,*

(b) $\langle \delta_{i^*}, \ldots, \delta_i \rangle$ *is a* γ*-coherent sequence for every* $i < n$,

(c) $\langle \delta_0, \ldots, \delta_n \rangle^\frown \langle \xi_0, \ldots, \xi_m \rangle$ *is a* γ*-coherent, where* $\langle \xi_0, \ldots, \xi_m \rangle$ *is* $0^{\vec{U}}(\delta_n)$-coherent,

(d) *if* $\gamma_1 > \gamma$, *then* $\langle \delta_0, \ldots, \delta_n \rangle$ *is* γ_1*-coherent.*

Notice that the 1-coherent sequences are exactly the finite increasing sequences of ordinals of $\vec{U}$-order 0.

The next lemma is obvious.

Lemma 3.3. *Let* $\beta_1 < \beta_2$ *be elements of* $\mathrm{dom}_1 \vec{U}$. *Suppose that* $\gamma \leqq \min(0^{\vec{U}}(\beta_1), 0^{\vec{U}}(\beta_2))$. *Then every* γ*-coherent sequence for* β_1 *is also* γ*-coherent for* β_2.

For every $\beta \in \mathrm{dom}_1 \vec{U} \cap (\alpha + 1)$, $\gamma \leqq 0^{\vec{U}}(\beta)$ and a γ-coherent sequence t for β, we shall define a forcing notion $\mathscr{P}(\beta, \gamma)$ and, for $\gamma < 0^{\vec{U}}(\beta)$, a set $U(\beta, \gamma, t)$ which, as will be shown later, will be a β-complete ultrafilter extending $U(\beta, \gamma)$.

Let us fix β. Suppose that $\langle \mathscr{P}(\beta, \gamma') \mid \gamma' < \gamma \rangle$ and $\langle U(\beta, \gamma', t) \mid \gamma' < \gamma$, t is a γ'-coherent sequence for $\beta \rangle$ are defined. Define first $\mathscr{P}(\beta, \gamma)$ and then, if $\gamma < 0^{\vec{U}}(\beta)$, $U(\beta, \gamma, t)$, for a γ-coherent sequence t.

Set $\mathscr{P}(\beta, 0) = \emptyset$. Define $\mathscr{P}(\beta, \gamma)$ for $\gamma > 0$.

For a γ-coherent sequence $\eta = \langle \tau_0, \ldots, \tau_{n-1} \rangle$ and $\gamma' < \gamma$, let us denote by $\eta(\gamma')$ the empty sequence, if $0^{\vec{U}}(\tau_{n-1}) \geqq \gamma'$ or, otherwise, the sequence $\langle \tau_i, \ldots, \tau_{n-1} \rangle$ for i the least s.t. for every j, $i \leqq j < n$, $0^{\vec{U}}(\tau_j) < \gamma'$. Clearly, $\eta(\gamma')$ is a γ'-coherent sequence.

If for some γ-coherent sequence η, some $\gamma' < \gamma$,

$$\{\delta < \beta \mid \eta^\frown \langle \delta \rangle \text{ is } \gamma\text{-coherent}\} \notin U(\beta, \gamma', \eta(\gamma')),$$

then set $\mathscr{P}(\beta, \gamma) = \emptyset$. Suppose otherwise, i.e. for every γ-coherent sequence η, every $\gamma' < \gamma$,

$$\{\delta < \beta \mid \eta^\frown \langle \delta \rangle \text{ is a } \gamma\text{-coherent sequence}\} \in U(\beta, \gamma', \eta(\gamma')).$$

Definition 3.4. A γ-tree is a tree consisting of γ-coherent sequences so that for every $\eta \in T$,

$$\mathrm{Suc}_T(\eta) = \bigcup_{\gamma' < \gamma} \mathrm{Suc}_{T,\gamma'}(\eta),$$

where $\mathrm{Suc}_{T,\gamma'}(\eta)$ is a set in $U(\beta, \gamma', \eta(\gamma'))$.

Remark. The tree of all γ-coherent sequences is a γ-tree.

For an element η of a γ-tree T denote

$$T_\eta = \{\nu \in T \mid \nu \text{ extends } \eta \text{ in } T\}.$$

DEFINITION 3.5. $\mathcal{P}(\beta, \gamma)$ is the set of all pairs $\langle t, T\rangle$, where t is a γ-coherent sequence and for some γ-tree T', so that $t \in T'$, $T = T'_t$.

DEFINITION 3.6. Let $\langle t_1, T_1\rangle$, $\langle t_2, T_2\rangle$ be in $\mathcal{P}(\beta, \gamma)$. Then $\langle t_2, T_2\rangle \geqq \langle t_1, T_1\rangle$ if there exists $\eta \in T_1$ so that

(a) $b_\eta = b_{t_2}$,

(b) for every $\nu \in T_2$

$$\eta^\frown\{\delta \in \nu \mid \delta > \max \eta\} \text{ is an element of } T_{1\eta}.$$

In case $t_1 = t_2$, let us call $\langle t_2, T_2\rangle$ a Prikry extension (or P-extension) of $\langle t_1, T_1\rangle$ and write $\langle t_2, T_2\rangle \geqq_P \langle t_1, T_1\rangle$.

Suppose that $\gamma < 0^{\vec{U}}(\beta)$. Define $U(\beta, \gamma, t)$ for a γ-coherent sequence t.

Let W be some fixed from the beginning, wellordering of V_λ s.t. for every inaccessible $\delta < \lambda$

$$tW \restriction V_\delta : V_\delta \leftrightarrow \delta,$$

where λ is a cardinal large enough.

Let $\langle \boldsymbol{A}_{\gamma'} \mid \gamma' < \beta^+\rangle$ be the $j_0^\beta(W)$-least enumeration of all canonical $\mathcal{P}_\beta$-names of subsets of β. Using the inductive assumptions on $\mathcal{P}j^\beta_\gamma(\beta)$ in N^β_γ, applying Lemmas 1.3 and 1.4, define a sequence of $\mathcal{P}_{\beta+1}$-names of elements of $\mathcal{P}_{j^\beta_\gamma(\beta)}/\mathcal{P}_{\beta+1}$, $\langle p_{\gamma'} \mid \gamma' < \beta^+\rangle$, so that

(1) for limit $\gamma' < \beta^+$

$$\| \boldsymbol{p}_{\gamma'} \text{ is the } j^\beta_\gamma(W)\text{-least E-extension for } \langle \boldsymbol{p}_{\gamma''} \mid \gamma'' < \gamma'\rangle \|^{\mathcal{P}_{\beta+1}} = 1,$$

(2) for every $\gamma' < \beta^+$

$$\| \boldsymbol{p}_{\gamma'+1} \text{ is the } j^\beta_\gamma(W)\text{-least E-extension of } \boldsymbol{p}_{\gamma'} \text{ deciding ``} \check{\beta} \in j^\beta_\gamma(\boldsymbol{A}_{\gamma'})\text{''} \|^{\mathcal{P}_{\beta+1}} = 1.$$

Now, for $A \subseteq \beta$, $A \in V[G \cap \mathcal{P}_\beta]$ set $A \in U(\beta, \gamma, t)$ if for some $r \in G \cap \mathcal{P}_\beta$, $\gamma' < \beta^+$, some name $\boldsymbol{A}$ of A and a $\mathcal{P}_\beta$-name $\boldsymbol{T}$ so that $r \cup \{\langle \check{t}, \boldsymbol{T}\rangle\} \in \mathcal{P}^{N^\beta_\gamma}_{\beta+1}$, in N^β_γ

$$r \cup \{\langle \check{t}, \boldsymbol{T}\rangle\} \cup \boldsymbol{p}_{\gamma'} \Vdash \check{\beta} \in j^\beta_\gamma(\boldsymbol{A}).$$

Suppose that for every $\beta \in \operatorname{dom}_1 \vec{U} \cap \alpha$ the following holds.

(A) For every $\gamma < 0^{\vec{U}}(\beta)$, every γ-coherent sequence t, $U(\beta, \gamma, t)$ is a β-complete ultrafilter extending $U(\beta, \gamma)$.

(B) For every $\gamma' < \gamma \leqq 0^{\vec{U}}(\beta)$, every γ-coherent sequence t,

$$\{\delta < \beta \mid t^\frown\langle\delta\rangle \text{ is a } \gamma\text{-coherent sequence}\} \in U(\beta, \gamma', t(\gamma')).$$

(C) For every $\gamma < 0^{\bar{U}}(\beta)$, every γ-coherent sequences t_1, t_2, if $b_{t_1} = b_{t_2}$, then $U(\beta, \gamma, t_1) = U(\beta, \gamma, t_2)$.

(D) $\mathcal{P}(\beta, 0^{\bar{U}}(\beta))$ is the forcing notion used on β.

(E) $\mathcal{P}(\beta, 0^{\bar{U}}(\beta))$ has the Prikry property, i.e. for every statement of the forcing language σ and every condition $\langle t, T\rangle$ there exists a Prikry extension of $\langle t, T\rangle$ deciding σ.

Let us prove (A)–(C), (E) for α and use the forcing $\mathcal{P}(\alpha, 0^{\bar{U}}(\alpha))$ over α.

LEMMA 3.7. *Let $\gamma < 0^{\bar{U}}(\alpha)$. Then*

(1) *for every $\gamma' < \gamma$ and a γ-coherent sequence t, $U(\alpha, \gamma', t)$ is equal to $U(\alpha, \gamma', t)$ in the sense of $N^\alpha_\gamma[G]$,*

(2) *$\mathcal{P}(\alpha, \gamma)$ is the forcing used on α, in $N^\alpha_\gamma[G]$.*

PROOF. (1) Let $j_{\delta\gamma}: N^\alpha_\delta \to N_{\delta\gamma} \simeq (N^\alpha_\delta)^{j^\alpha_\beta(\alpha)} / j^\alpha_\delta(U(\alpha, \gamma))$ and $j_{\gamma\delta}: N^\alpha_\gamma \to N_{\gamma\delta} \simeq (N^\alpha_\gamma)^\alpha / U(\alpha, \delta)$ be the canonical elementary embeddings, for $\delta < \gamma < 0^{\bar{U}}(\alpha)$.

Then $N_{\gamma\delta} = N_{\delta\gamma} =_{\mathrm{df}} N$ and the following diagram is commutative.

$$\begin{array}{ccc} V & \xrightarrow{\quad j_\delta \quad} & N^\alpha_\delta \\ {\scriptstyle j_\gamma}\downarrow & & \downarrow{\scriptstyle j_{\delta\gamma}} \\ N^\alpha_\gamma & \xrightarrow{\quad j_{\gamma\delta} \quad} & N \end{array}$$

The first ordinal moved by $j_{\delta\gamma}$ is $j^\alpha_\delta(\alpha)$. $N^\alpha_\delta \cap {}^{j^\alpha_\gamma(\alpha)}N \subseteq N^\alpha_\delta$. Hence

$$j_{\gamma\delta}(j^\alpha_\gamma(W)) \mid V^N_{j^\alpha_\delta(\alpha)} = j_{\delta\gamma}(j_\delta(W)) \mid V^N_{j^\alpha_\delta(\alpha)} = j_\delta(W) \mid V^{N^\alpha_\delta}_{j^\alpha_\delta(\alpha)}.$$

Applying this for $\delta = 0$, we obtain that the minimal sequence $\langle \boldsymbol{A}_{\gamma'} \mid \gamma' < \alpha^+\rangle \in N^\alpha_0$ is also such for N.

Now let $\delta = \gamma'$. The forcing $\mathcal{P}_{j^\alpha_{\gamma'}(\alpha)}$ is the same in $N^\alpha_{\gamma'}$, N^α_γ since $j^\alpha_{\gamma'}(\alpha) = j_{\gamma\gamma'}(\alpha)$ and

$$\mathcal{P}_{j^\alpha_{\gamma'}(\alpha)} \subseteq V^{N_{\gamma'}}_{j^\alpha_{\gamma'}(\alpha)} = V^N_{j^\alpha_{\gamma'}(\alpha)}.$$

Hence $U(\alpha, \gamma', t)$ in the sense ot $N^\alpha_\gamma[G]$ is $U(\alpha, \gamma', t)$.

(2) It follows from (1) and (C). □

Lemma 3.7 and (A), (B), and (C) for $\beta < \alpha$ imply (A), (B) and (C) for every $\gamma < 0^{\bar{U}}(\alpha)$ so that $\gamma + 1 < 0^{\bar{U}}(\alpha)$. So (A)–(C) holds if $0^{\bar{U}}(\alpha)$ is a limit ordinal. Suppose now that $0^{\bar{U}}(\alpha) = \gamma + 1$ for some γ.

LEMMA 3.8. *For every γ-coherent sequence t, $U(\alpha, \gamma, t)$ is an α-complete ultrafilter on α extending $U(\alpha, \gamma)$.*

PROOF. By Lemma 3.7 (2), the forcing on α, in $N^\alpha_\gamma[G]$, is $\mathscr{P}(\alpha,\gamma)$. (D) in $N^\alpha_\gamma[G]$ implies that $\mathscr{P}(\alpha,\gamma)$ has the Prikry property. The inductive assumptions (A), (B) and (C) in $N^\alpha_\gamma[G]$ imply that $\mathscr{P}(\alpha,\gamma)$ is a α-weakly closed forcing and, even more, if $\langle\langle t, T_\xi\rangle \mid \xi<\xi'<\alpha\rangle$ are conditions in $\mathscr{P}(\alpha,\gamma)$ then $\langle t, \bigcap_{\xi<\xi'} T_\xi\rangle$ is also a condition in $\mathscr{P}(\alpha,\gamma)$.

Now the proof is almost the same as the proof of Lemma 2.2. Just replace the membership in $\bar{U}(\kappa,0)$ by being a γ-tree. □

It is not hard to see that $\bigcap\{U(\alpha,\gamma,t) \mid t$ is a γ-coherent sequence$\}$ is a normal α^+-saturated filter on α.

Notice, also, that if $\gamma=0$, then $U(\alpha,0,\langle\ \rangle)$ is a normal ultrafilter. So we have proved (A) for α. Let us show (B). There remains only one case.

LEMMA 3.9. *For every $\gamma+1$-coherent sequence $t^\frown\{\delta<\alpha \mid t^\frown\langle\delta\rangle$ is a $\gamma+1$-coherent sequence$\}\in U(\alpha,\gamma,t(\gamma))$.*

PROOF. Notice that the nontrivial case is when the $\vec{U}$-order of the maximal element of t is less than γ, since otherwise every δ with $0^{\vec{U}}(\delta)=\gamma$ will be O.K.

Suppose $r\in G$ forces "$t(\check{\gamma})$ is $\check{\gamma}$-coherent". Then, in N_γ, $r\Vdash_{\mathscr{P}_\alpha}$ "$t(\check{\gamma})$ is $\check{\gamma}$-coherent" and, if $0^{\vec{U}}(\alpha)=\gamma+1>1$, then $r\cup\{\langle t(\gamma),\boldsymbol{T}\rangle\}\Vdash_{\mathscr{P}_{\alpha+1}}$ "$t(\check{\gamma})^\frown\langle\check{\alpha}\rangle$ is $\check{\gamma}+1$-coherent", for some $\mathscr{P}_\alpha$-name $\boldsymbol{T}$ of a γ-tree with trunk $t(\gamma)$.

But then, by the definition of $U(\alpha,\gamma,t(\gamma))$, the set $A=\{\delta<\alpha \mid 0^{\vec{U}}(\delta)=\gamma$ and $t(\gamma)^\frown\langle\delta\rangle$ is $\gamma+1$-coherent sequence$\}\in U(\alpha,\gamma,t(\gamma))$. So, $t^\frown\langle\delta\rangle$ is $\gamma+1$-coherent for every $\delta\in A$, since the maximal $\xi\in t-t(\gamma)$ (if there is any) has $\vec{U}$-order γ.

□

Let us prove (C). As above, it is enough to prove the following.

LEMMA 3.10. *For every γ-coherent sequence t_1, t_2 if $b_{t_1}=b_{t_2}$, then $U(\alpha,\gamma,t_1)=U(\alpha,\gamma,t_2)$.*

PROOF. Let $A\subseteq\alpha$ be in $V[G]$. If $A\in U(\alpha,\gamma,t_i)$ $(i=1,2)$, then for some $r\in G$, $\gamma'<\alpha^+$ and a $\mathscr{P}_\alpha$-name $\boldsymbol{T}_i$ so that $r\cup\{\langle\check{t}_i,\boldsymbol{T}_i\rangle\}\in\mathscr{P}^{N^\beta_\gamma}_{\alpha+1}$, in N^α_γ $r\cup\{\langle\check{t}_i,T_i\rangle\}\cup\boldsymbol{P}_{\gamma'}\Vdash\check{\alpha}\in j^\alpha_\gamma(A)$. Let $A\in U(\alpha,\gamma,t_1)$. Pick some $r'\geqq r$, $r'\in G$ forces "$b_{t_1}=b_{t_2}$". (D) in N^α_γ implies that the forcing used on α is $\mathscr{P}(\alpha,\gamma)$. Set

$$T_2=\{t_2{}^\frown\{\delta\in\nu \mid \delta>\max t_2\} \mid \nu\in T_1\}.$$

Then T_2 is a γ-tree. Since for every $\delta<\gamma$, $U(\alpha,\delta,t_1(\delta))=U(\alpha,\delta,t_2(\delta))$, by the inductive assumption (C), in N^α_γ. Notice that $b_{t_1(\delta)}=b_{t_2(\delta)}$, since by (D) for $\beta<\alpha$ b_β consists of ξ's with $0^{\vec{U}}(\xi)<0^{\vec{U}}(\beta)$.

Hence $\langle t_2,T_2\rangle$ is a condition in $\mathscr{P}(\alpha,\gamma)$. But its strength is the same as $\langle t_1,T_1\rangle$, i.e. $\langle t_2,T_2\rangle\geqq\langle t_1,T_1\rangle$ and $\langle t_1,T_1\rangle\geqq\langle t_2,T_2\rangle$.

Then

$$r' \cup \{\langle \check{t}_2, \boldsymbol{T}_2\rangle\} \cup p_{\gamma'} \Vdash \check{\alpha} \in j^{\alpha}_{\gamma}(\boldsymbol{A}).$$

So $A \in U(\alpha, \gamma, t_2)$. Hence $U(\alpha, \gamma, t_2) \supseteq U(\alpha, \gamma, t_1)$, but they are ultrafilters. So $U(\alpha, \gamma, t_1) = U(\alpha, \gamma, t_2)$. □

It remains to prove (E).

LEMMA 3.11. *$\mathscr{P}(\alpha, 0^{\bar{U}}(\alpha))$ has the Prikry property.*

PROOF. Let $\langle t, T\rangle \in \mathscr{P}(\alpha, 0^{\bar{U}}(\alpha))$ and σ be a statement of the forcing language.

For $\eta \in T$, $\delta < \gamma$ w.l.o.g., let every $\beta \in \mathrm{Suc}_{T,\delta}(\eta)$ have $\bar{U}$-order δ, where as in 3.4

$$\mathrm{Suc}_T(\eta) = \bigcup_{\delta<\gamma} \mathrm{Suc}_{t,\delta}(\eta).$$

So every $\mathrm{Suc}_{T,\delta}(\eta) \in U(\alpha, \delta, \eta(\delta))$. Each $U(\alpha, \gamma, t)$ is an α-complete ultrafiler on α.

Let us shrink T level by level to an $0^{\bar{U}}(\alpha)$-tree T^* with the trunk t so that for every $\nu \in T^*$, $\delta < 0^{\bar{U}}(\alpha)$, $\beta \in \mathrm{Suc}_{T^*,\delta}(\nu)$ if for some T'

$$\langle \nu^\frown\langle\beta\rangle, T'\rangle \Vdash {}^i\sigma$$

then

$$\langle \nu^\frown\langle\beta\rangle, T_{\nu^\frown\langle\beta\rangle}\rangle \Vdash {}^i\sigma$$

and for every $\beta' \in \mathrm{Suc}_{T^*,\delta}(\nu)$

$$\langle \nu^\frown\langle\beta'\rangle, T_{\nu^\frown\langle\beta'\rangle}\rangle \Vdash {}^i\sigma$$

where $i \in 2$, ${}^0\sigma \equiv \sigma$ ${}^1\sigma \equiv \neg\sigma$ and

$$T^*_{\nu^\frown\langle\beta\rangle} = \{\nu' \in T^* \mid \nu' \text{ extends } \nu^\frown\langle\beta\rangle\}.$$

We claim that $\langle t, T^*\rangle$ decides σ. Otherwise, some $\langle t'_i, T'_i\rangle \geqq \langle t, T^*\rangle$ forces ${}^i\sigma$ ($i \in 2$). Let us pick such $\langle t'_i, T'_i\rangle$ so that $|t'_i - t|$ is the minimal possible. We are going to show that $t'_i = t$. It will imply the contradiction, since then $\langle t, T'_1 \cap T'_0\rangle$ is in $\mathscr{P}(\alpha, 0^{\bar{U}}(\alpha))$ and is stronger than both $\langle t'_1, T'_1\rangle$ and $\langle t'_0, T'_0\rangle$.

Let us prove that $t'_0 = t$. The proof of $t'_1 = t$ is similar. Suppose that $t'_0 \neq t$. Then $t'_0 = t^\frown\nu^\frown\langle\beta\rangle$. Assume for simplicity that $\nu = \langle\ \rangle$. Let $\beta \in \mathrm{Suc}_{T^*,\delta}(t)$ for some $\delta < 0^{\bar{U}}(\alpha)$. Then

$$\langle t^\frown\langle\beta\rangle, T^*_{t^\frown\langle\beta\rangle}\rangle \Vdash \sigma$$

and so for every $\beta' \in \mathrm{Suc}_{T^*,\delta}(t)$

$$\langle t^\frown\langle\beta'\rangle, T^*_{t^\frown\langle\beta'\rangle}\rangle \Vdash \sigma.$$

CLAIM 3.11.1. *For every* δ', $0^{\bar U}(\alpha) > \delta' \geqq \delta$, *every* $\beta' \in \mathrm{Suc}_{T^*,\delta'}(t)$, $\langle t^\frown\langle\beta'\rangle, T^*_{t^\frown\langle\beta'\rangle}\rangle \Vdash \sigma$.

PROOF. Fix δ', $\delta < \delta' < 0^{\bar U}(\alpha)$. It is enough to find one $\beta' \in \mathrm{Suc}_{T^*,\delta'}(t)$ s.t.

$$\langle t^\frown\langle\beta'\rangle, T^*_{t^\frown\langle\beta'\rangle}\rangle \Vdash \sigma.$$

Let $A = \mathrm{Suc}_{T^*,\delta'}(t)$. Then $A \in U(\alpha, \delta', t(\delta'))$. So for some $r \in G$, a δ'-tree T with trunk $t(\delta')$ and $\gamma' < \alpha^+$, in $N^\alpha_{\gamma'}[G]$

$$r \cup \{\langle \check t(\delta'), \boldsymbol{T}\rangle\} \cup \boldsymbol{p}_{\gamma'} \Vdash \check\alpha \in j^\alpha_{\delta'}(\boldsymbol{A}).$$

Notice than the forcing on α in $N^\alpha_{\delta'}[G]$ is $\mathcal{P}(\alpha, \delta')$.

Since $\delta' > \delta$, $\mathrm{Suc}_{T,\delta}(t(\delta')) \in U(\alpha, \delta, t(\delta))$. Hence there exists some $\beta \in \mathrm{Suc}_{T,\delta}(t(\delta')) \cap \mathrm{Suc}_{T^*,\delta}(t)$. Then $\langle t(\delta')^\frown\langle\beta\rangle, T_{t(\delta')^\frown\langle\beta\rangle}\rangle \geqq \langle t(\delta'), T\rangle$ and hence, in $N^\alpha_{\delta'}[G]$ for some $r' \in G$, $r' \geqq r$,

$$r' \cup \{\langle \check t(\delta')^\frown\langle\beta\rangle, \boldsymbol{T}_{t(\delta')^\frown\langle\beta\rangle}\rangle\} \cup \boldsymbol{p}_\gamma \Vdash \check\alpha \in j^\alpha_{\delta'}(\boldsymbol{A}).$$

So $A \in U(\alpha, \delta', t(\delta')^\frown\langle\beta\rangle)$. Let, then, $\beta' \in A \cap \mathrm{Suc}_{T^*,\delta'}(t^\frown\langle\beta\rangle)$. Clearly, there is such since $(t^\frown\langle\beta\rangle)(\delta') = t(\delta')^\frown\langle\beta\rangle$. The condition $\langle t^\frown\langle\beta,\beta'\rangle, T^*_{t^\frown\langle\beta,\beta'\rangle}\rangle$ is stronger than the condition $\langle t^\frown\langle\beta\rangle, T^*_{t^\frown\langle\beta\rangle}\rangle$ forcing σ. On the other hand, $\beta \in b_{\beta'}$. So $b_{t^\frown\langle\beta'\rangle} = b_{t^\frown\langle\beta,\beta'\rangle}$. (C) implies then that for every $\delta'' < 0^{\bar U}(\alpha)$,

$$U(\alpha, \delta'', (t^\frown\langle\beta'\rangle)(\delta'')) = U(\alpha, \delta'', (t^\frown\langle\beta,\beta'\rangle)(\delta'')),$$

since, clearly, $b_{(t^\frown\langle\beta'\rangle)(\delta'')} = b_{(t^\frown\langle\beta,\beta'\rangle)(\delta'')}$. Define $T_1 = \{\nu - \langle\beta\rangle \mid \nu \in T^*_{t^\frown\langle\beta,\beta'\rangle}\}$. Then T_1 is an $0^{\bar U}(\alpha)$-tree. So $\langle t^\frown\langle\beta'\rangle, T_1\rangle \in \mathcal{P}(\alpha, 0^{\bar U}(\alpha))$ and it is the same as $\langle t^\frown\langle\beta,\beta'\rangle, T_{t^\frown\langle\beta,\beta'\rangle}\rangle$. Hence, $\langle t^\frown\langle\beta'\rangle, T_1\rangle$ forces σ. Then, also, $\langle t^\frown\langle\beta'\rangle, T_1 \cap T^*_{t^\frown\langle\beta'\rangle}\rangle$ forces σ. So $\langle t^\frown\langle\beta'\rangle, T^*_{t^\frown\langle\beta'\rangle}\rangle$ forces σ, by the choice of T^*. □ of the claim

CLAIM 3.11.2. *For every* $\delta' < \delta$, *every* $\beta' \in \mathrm{Suc}_{T^*,\delta'}(t)$, $\langle t^\frown\langle\beta'\rangle, T^*_{t^\frown\langle\beta'\rangle}\rangle \Vdash \sigma$.

PROOF. Fix $\delta' < \delta$. It is enough to find some $\beta' \in \mathrm{Suc}_{T^*,\delta'}(t)$ s.t. $\langle t^\frown\langle\beta'\rangle, T^*_{t^\frown\langle\beta'\rangle}\rangle \Vdash \sigma$.

Let $B = \mathrm{Suc}_{T^*,\delta}(t)$. Then for some $r \in G$, δ-tree T with trunk $t(\delta)$ and $\gamma' < \alpha^+$, in $N^\alpha_\delta[G]$

$$r \cup \{\langle \check t(\delta), \boldsymbol{T}\rangle\} \cup \boldsymbol{p}_{\gamma'} \Vdash \check\alpha \in j^\alpha_\delta(\boldsymbol{B}).$$

Let us define an $0^{\bar U}(\alpha)$-tree T^{**} with a trunk t which will be a natural intersection of T^* with T.

Set $T^{**}=\{t^\frown\langle\tau_0,\dots,\tau_{n-1}\rangle\in T^*\mid$ if $i<n$ is maximal s.t. for every $j\leqq i$ $0^{\bar U}(\tau_j)<\delta$, then $t(\delta)^\frown\langle\tau_0,\dots,\tau_i\rangle\in T\}$.

Let $\beta'\in \mathrm{Suc}_{T^{**},\delta'}(t)$. Suppose that $\langle t^\frown\langle\beta'\rangle, T^{**}_{t^\frown\langle\beta'\rangle}\rangle\not\Vdash\sigma$. Pick $\langle t^\frown\langle\beta',\tau_0,\dots,\tau_{n-1}\rangle, T_1\rangle\geqq\langle t^\frown\langle\beta'\rangle, T^{**}_{t^\frown\langle\beta'\rangle}$ forces $\neg\sigma$. W.l.o.g. $T_1=T^*_{t^\frown\langle\beta',\tau_0,\dots,\tau_{n-1}\rangle}$. Let $i<n$ be maximal s.t. for every $j\leqq i$, $0^U(\tau_j)<\delta$. Then $t^*=t(\delta)^\frown\langle\beta',\tau_0,\dots,\tau_i\rangle\in T$. Pick some $r'\geqq r$, $r'\in G$ forces $\langle t^*, T_{t^*}\rangle\in\mathcal{P}(\alpha,\delta)$. Then

$$r'\cup\{\langle\check{t}^*, \boldsymbol{T}_{t^*}\rangle\}\cup \boldsymbol{p}'_\gamma\Vdash\check\alpha\in j^\alpha_\delta(\boldsymbol{B}).$$

Hence $B\in U(\alpha,\delta,t^*)$. Pick some $\beta\in B\cap \mathrm{Suc}_{T_1,\delta}(t^\frown\langle\beta',\tau_0,\dots,\tau_i\rangle)$. Then $b_{t^\frown\langle\beta\rangle}=b_{t^\frown\langle\beta',\tau_0,\dots,\tau_i\rangle^\frown\langle\beta\rangle}$ and for every $\delta''<0^{\bar U}(\alpha)$

$$b_{(t^\frown\langle\beta\rangle)(\delta'')}=b_{(t^\frown\langle\beta',\tau_0,\dots,\tau_i\rangle^\frown\langle\beta\rangle)(\delta'')}.$$

Define $T_2=\{\nu-\{\langle\beta',\tau_0,\dots,\tau_i\rangle\}\mid\nu\in T^*_{t^\frown\langle\beta',\tau_0,\dots,\tau_i,\beta\rangle}\}$,

$$T_2=\{\nu^\frown\langle\beta',\tau_0,\dots,\tau_i\rangle\mid\nu\in T^*_{t^\frown\langle\beta\rangle}\}.$$

(C) implies that T_2 is an $0^{\bar U}(\alpha)$-tree with a trunk $t^\frown\langle\beta',\tau_0,\dots,\tau_i,\beta\rangle$. Then $\langle t^\frown\langle\beta',\tau_0,\dots,\tau_i,\beta\rangle, T_2\rangle$ is equivalent to $\langle t^\frown\langle\beta\rangle, T^*_{t^\frown\langle\beta\rangle}\rangle$ in $\mathcal{P}(\alpha, 0^{\bar U}(\alpha))$. Hence

$$\langle t^\frown\langle\beta',\tau_0,\dots,\tau_i,\beta\rangle, T_2\rangle\Vdash\sigma.$$

So also $\langle t^\frown\langle\beta',\tau_0,\dots,\tau_i,\beta\rangle,\ T_2\cap T^*_{t^\frown\langle\beta',\tau_0,\dots,\tau_i,\beta\rangle}\rangle\Vdash\sigma$.

By Claim 3.11.1, we can replace β by any $\beta''\in\mathrm{Suc}_{T^*,\delta''}(t^\frown\langle\beta',\tau_0,\dots,\tau_i\rangle)$ for $\delta''\geqq\delta$. But it is impossible, since $\langle t^\frown\langle\beta',\tau_0,\dots,\tau_{n-1}\rangle,\ T^*_{t^\frown\langle\beta',\tau_0,\dots,\tau_{n-1}\rangle}\rangle\Vdash\neg\sigma$ and, if $i<n-1$, $0^{\bar U}(\tau_{i+1})\geqq\delta$. Contradiction. □ of the claim

□

This completes the proof of conditions (A)–(D) for α. We defined Q_α to be $\mathcal{P}(\alpha, 0^{\bar U}(\alpha))$. Set $\mathcal{P}_{\alpha+1}=\mathcal{P}_\alpha * \boldsymbol{Q}_\alpha$. For every $\beta\leqq 0^{\bar U}(\alpha)$, the forcing notion $\mathcal{P}(\alpha,\beta)$ has the Prikry property and it is an α-weakly closed. So if $\beta\leqq 0^{\bar U}(\alpha)$ is a cardinal (it does not matter whether in V or in $V[G]$ since they have the same cardinals), then the forcing with $\mathcal{P}(\alpha,\beta)$ over $V[G]$ changes the cofinality of α to $\mathrm{cof}^{V[G]}(\beta)$ without adding new bounded subsets to α.

Mitchell [Mi 2] constructed a model with α^+-saturated normal filter over α concentrating on singular cardinals of cofinality δ (for $\delta<\alpha$). Such filters can be defined in $V[G]$, provided that δ is a regular cardinal of V, $\delta\notin\mathrm{dom}_1\bar U$ and $0^{\bar U}(\alpha)>\delta$.

$$\mathcal{F}=\bigcap\{U(\alpha,\delta,t)\mid t \text{ is a } \delta\text{-coherent sequence}\}$$

will be such a filter. In case $\delta=\omega$ just $0^{\bar U}(\alpha)>1$ is enough, since $\mathcal{F}=$

$\bigcap\{U(\alpha,1,t) \mid t$ is a 1-coherent sequence$\}$ is α^+-saturated concentrating on cofinality ω.

If $0^{\vec{U}}(\alpha)=\alpha$, then we can define $\mathcal{F}=\bigcap\{U(\alpha,\gamma,t) \mid \gamma<0^{\vec{U}}(\alpha)$, t is a γ-coherent sequence$\}$. $\mathcal{F}$ will be α^+-saturated normal filter and for every $\beta<\alpha$ the set $\{\delta<\alpha \mid \mathrm{cof}^{V[G]}(\delta)=\mathrm{cof}^{V[G]}(\beta)\}$ will be $\mathcal{F}$-positive.

§4. The precipitousness of the nonstationary ideal

Let us describe first how to construct a model with NS_κ (the non-stationary ideal over κ) precipitous over an inaccessible κ. Start with a measurable cardinal κ of the Mitchell order κ. Pick a coherent sequence $\vec{U}$ of length $l^{\vec{U}}=\kappa+1$ and with $0^{\vec{U}}(\kappa)=\kappa$. Let $\mathcal{P}_\kappa$ be the forcing notion defined in §3 for $\vec{U}$. Let G be a generic subset of $\mathcal{P}_\kappa$. Then every $A\in\bigcap_{\alpha<\kappa}U(\kappa,\alpha)$ will be a fat subset of κ in $V[G]$ (i.e., for every club $C\subseteq\kappa$, $A\cap C$ contains closed sets of ordinals of arbitrarily large order-types below κ). So, by Avraham–Shelah [A–S] it is possible to shoot a club through A without adding new bounded subsets of κ. The problem starts when we try to iterate such forcing. We shall apply ideas of [G 2]. We need to have a generic club through $A\cap\alpha$ inside $V[G]$, for a lot of α's below κ. In order to obtain it, we shall change cofinalities of both α, α^+ to ω.

In [G 2] we used a variant of the Namba forcing. Here we like to preserve cardinals. A strongly compact Prikry forcing will be used.

Assume G.C.H. Let $\vec{U}$ be a coherent sequence as above. Assume, in addition, that $U(\kappa,0)$ concentrates on α^{++}-strongly compact cardinals α. For $U(\kappa,0)$ concentrating on α^+-super-compact cardinals the proof we are giving works as well. Let us change slightly the definition of $\mathcal{P}_\kappa$ of §3. Let

$$A=\{\alpha<\kappa \mid \alpha\notin \mathrm{dom}_1\vec{U},\ \alpha \text{ is an } \alpha^{++}\text{-strongly compact cardinal}\}.$$

Then $A\in U(\kappa,0)$. W.l.o.g., we can assume that for every $\alpha\in\mathrm{dom}_1\vec{U}$, $A\cap\alpha\in U(\alpha,0)$. To obtain it just shrink the domain of $\vec{U}$ enough. For every $\alpha\in A$ fix some fine measure $U(\alpha)$ over $\mathcal{P}_\alpha(\alpha^{++})$ so that in the ultrapower by $U(\alpha)$, $|[\mathrm{id}]_{U(\alpha)}|=\alpha^{++}$.

We shall use the forcing of the kind of §§1–3 for the coherent sequence $\vec{U}$. Just, in addition, let us change cofinalities of every α, α^+ to ω for α in A. So we define by induction the iteration $\mathcal{P}_\alpha$ for $\alpha\in$ the closure of $\{\beta \mid \beta\in A\cup\mathrm{dom}_1\vec{U}$ or $\beta=\gamma+1$ and $\gamma\in A\cup\mathrm{dom}_1\vec{U}\}$. Suppose that $\mathcal{P}_\alpha$ is defined and $\alpha\in A$. Let $\langle A_\beta \mid \beta<\alpha^{++}\rangle$ be the W-least enumeration of all canonical $\mathcal{P}_\alpha$-names of subsets of $\mathcal{P}^V_\alpha(\alpha^+)$. Let

$$j\colon V\to N\simeq V^{\mathcal{P}_\alpha(\alpha^{++})}/U(\alpha)$$

be the canonical elementary embedding. In N there exists a set B cardinality $\alpha^{++}=|[\mathrm{id}]|$ so that for every $\beta<\alpha^{++}$, $j(A_\beta)\in B$.

Pick in $N^{\mathcal{P}_{\alpha+1}}$ the $j(W)$-least $\mathcal{P}_{\alpha+1}$-name of $p\in\mathcal{P}_{j(\alpha)}/\mathcal{P}_{\alpha+1}$ deciding all the statements "$j(\check{\alpha}^+)\cap[\mathrm{id}]\in b$" for b in B. It exists by Lemma 1.2. In case α α^+-supercompact, define an α^{++}-sequence of $\mathcal{P}_{\alpha+1}$-names of elements of $\mathcal{P}_{j(\alpha)}/\mathcal{P}_\alpha$ deciding all the statements $j''(\alpha^+)\in j(A_\beta)$ $(\beta<\alpha^{++})$, as was done in Lemmas 1.5 and 2.2.

Let G be a generic subset of $\mathcal{P}_\alpha$. Define now a fine ultrafilter $\bar{U}(\alpha)$ over $\mathcal{P}_\alpha(\alpha^+)$ in $V[G]$ as follows. Set $C\in\bar{U}(\alpha)$, if $C\subseteq\mathcal{P}_\alpha(\alpha^+)$ and, in $N^{\mathcal{P}_\alpha}$,

$$\|p\Vdash j(\check{\alpha}^+)\cap[\mathrm{id}]_{U(\alpha)}\in j(C)\|^{\mathcal{P}_\alpha}\in G,$$

in case $\mathcal{P}_{\alpha+1}/\mathcal{P}_\alpha=\emptyset$, or if $\mathcal{P}_{\alpha+1}/\mathcal{P}_\alpha\neq\emptyset$ and for some T s.t. $\langle\emptyset,T\rangle\in\mathcal{P}_{\alpha+1}/\mathcal{P}_\alpha$,

$$\|\{\langle\emptyset,T\rangle\}\cup p\Vdash j(\check{\alpha}^+)\cap[\mathrm{id}]_{U(\alpha)}\in j(C)\|^{\mathcal{P}_\alpha}\in G.$$

The checking that such defined $\bar{U}(\alpha)$ is a fine ultrafilter is as in Lemmas 1.5 and 2.2. Now let Q_α be the strongly compact Prikry forcing with $\bar{U}(\alpha)$. Conditions in Q_α are of the form $\langle Q_1,\dots,Q_n,B\rangle$, where $\langle Q_1,\dots,Q_n\rangle$ is an increasing sequence of elements of $\mathcal{P}_\alpha(\alpha^+)$ and B is a tree of increasing sequences of elements of $\mathcal{P}_\alpha(\alpha^+)$ so that Q_n is contained in every element of such a sequence and for every $\eta\in B$, $\mathrm{Suc}_B(\eta)\in\bar{U}(\alpha)$. The forcing notion Q_α is α-weakly closed, has the Prikry property, and changes cofinalities of both α and α^+ to ω. We refer to [P], [G 1] for detailed information on such forcing notions.

Define $\mathcal{P}_{\alpha+1}=\mathcal{P}_\alpha * \boldsymbol{Q}_\alpha$.

Now, for $\alpha\in\mathrm{dom}_1\vec{U}$, define $\mathcal{P}_{\alpha+1}$ as in §3; just to extend $U(\alpha,0)$ use conditions in the strongly compact Prikry forcing with empty finite sequence. This extension will no longer be a normal ultrafilter, but its normality was not used in §3.

Fix now a generic subset G_κ of $\mathcal{P}_\kappa$. Set $G_\alpha=G_\kappa\cap\mathcal{P}_\alpha$. Then every G_α will be a V-generic subset of $\mathcal{P}_\alpha$. Let $E\in\bigcap_{\beta<\kappa}U(\kappa,\beta)$. Set for $\gamma<\kappa$

$$E(\gamma)=\left\{\beta\in E\;\middle|\;0^{\vec{U}}(\beta)\geqq\gamma\text{ and }E\cap\beta\in\bigcap_{\sigma<\gamma}U(\beta,\sigma)\right\}.$$

For every $\beta\in E(\gamma)$, b_β, the generic sequence to β, is almost contained in $E\cap\beta$. The order type of b_β is $\geqq\gamma$. The set $E(\gamma)$ is in $U(\kappa,\gamma)$ and, in particular, it is stationary. Notice that, since $\mathcal{P}_\kappa$ satisfies κ-c.c., by Baumgartner [B] every club of κ of $V[G_\kappa]$ contains a club of κ of V. So, if a set is a stationary subset of κ in V it is still stationary in $V[G_\kappa]$. The same holds if we replace κ by $\alpha\in A\cup\mathrm{dom}_1\vec{U}$.

Set $E' = \{\alpha < \kappa \mid \alpha$ is a measurable in V for every $\gamma < \alpha$, $E(\gamma) \cap \alpha$ is a stationary subset of $\alpha\}$.

Then $E' \in \bigcap_{\beta<\kappa} U(\kappa, \beta)$. It is not hard to see that for every $\alpha \in E' \cup \{\kappa\}$, $E \cap \alpha$ is a fat subset of α in $V[G]$, i.e. for every club $C \subseteq \alpha$, $E \cap C$ contains closed sets of ordinals of arbitrarily large order-type below α. Define by induction the following sets:

$$E_0 = \{\alpha \in E \cap E' \mid \alpha \in A\}, \qquad \text{for } \beta, \quad 1 \leqq \beta \leqq \kappa,$$

$$E_\beta = \{\alpha \in E \cap E' \mid 0^{\bar{U}}(\alpha) = \beta \text{ and for every } \delta < \beta,\, E_\delta \cap \alpha \in U(\alpha, \delta)\}.$$

Then each E_β is in $U(\kappa, \beta)$. Set $E^{(1)} = U\{E_\beta \mid \beta < \kappa\}$.

Let $\alpha \in E^{(1)} \cup \{\kappa\}$. Define $P[E \cap \alpha]$ to be the forcing notion in $V[G_\alpha]$ consisting of all closed subsets of $E \cap \alpha$ and ordered by endextension.

LEMMA 4.1. *For every $\alpha \in E^{(1)}$ there exists a $V[G_\alpha]$-generic subset of $P[E \cap \alpha]$ in $V[G_{\alpha+1}]$.*

PROOF. If $\alpha \in E_0$, then the strongly compact Prikry forcing was used on α and $\operatorname{cof} \alpha^+ = \operatorname{cof} \alpha = \omega$ in $V[G_{\alpha+1}]$. Also, there is no new bounded subsets of α in $V[G_{\alpha+1}]$. Since $E \cap \alpha$ is a fat subset of α, the forcing $P[E \cap \alpha]$ is an (α, ∞)-distributive forcing notion in $V[G_\alpha]$; see [A–S]. Now the set of all dense subsets of $P[E \cap \alpha]$ which are in $V[G_\alpha]$ can be written in $V[G_{\alpha+1}]$ as a union of ω sets; each of them is in $V[G_\alpha]$ and has cardinality less than α. Then using the distributivity of $P[E \cap \alpha]$, we can define a set meeting all dense subsets of $P[E \cap \alpha]$ of $V[G_\alpha]$.

Suppose now that for every $\gamma < \beta < \kappa$, $\alpha \in E_\gamma$, the lemma is proved. Let us prove it for β. So, let $\alpha \in E_\beta$. We would like to find a $V[G_\alpha]$-generic subset of $P[E \cap \alpha]$ in $V[G_{\alpha+1}]$. Consider the generic sequence b_α to α. For some $\xi < \alpha$, $b_\alpha - \xi$ will be contained in every $E_\delta \cap \alpha$ $(\delta < \beta)$. Denote $b_\alpha - \xi$ by b. Let $\langle \gamma_\xi \mid \xi < \tau \rangle$ be the increasing continuous enumeration of b. Define by induction an increasing continuous sequence of closed sets $\langle C_\xi \mid \xi < \tau \rangle$ so that

(1) $C_\xi \in V[G_{\gamma_\xi+1}]$ is a $V[G_{\gamma_\xi}]$-generic club through $E \cap \gamma_\xi$,

(2) $C_{\xi+1}$ is an endextension of C_ξ,

(3) for a limit ξ, $C_\xi = U\{C_{\xi'} \mid \xi' < \xi\}$.

Define $C_{\xi+1}$ using the inductive assumption. Let us show for a limit ξ, $C_\xi = U\{C_{\xi'} \mid \xi' < \xi\}$ is $V[G_{\gamma_\xi}]$-generic.

Let D be a dense open subset of $P[E \cap \gamma_\xi]$ in $V[G_{\gamma_\xi}]$. Then for some $\xi' < \xi$, $D \cap P[E \cap \gamma_{\xi'}]$ will belong to $V[G_{\gamma_{\xi'}}]$ and it will be a dense subset of $P[E \cap \gamma_{\xi'}]$ in $V[G_{\gamma_{\xi'}}]$. Since $\mathcal{P}_{\gamma_\xi}$ satisfies γ_ξ-c.c., $\langle \gamma_{\xi''} \mid \xi'' < \xi \rangle$ is almost contained in every club of γ_ξ and for $\delta < \gamma_\xi$ the forcing $\mathscr{P}_{\gamma_\xi}/\mathscr{P}_\delta$ does not add new bounded subsets to δ. The $C_{\gamma_{\xi'}}$ extends some element of $D \cap P[E \cap \gamma_\xi]$. Hence $C_{\gamma_{\xi'}} \cup \{\gamma_\xi\}$ belongs to D. So also C_{γ_ξ} extends an element of D. Hence C_{γ_ξ} is a $V[G_{\gamma_\xi}]$-generic club. It completes the induction. □

$E^{(1)} \in \bigcap_{\beta<\kappa} U(\kappa,\beta)$. Replace now E by $E^{(1)}$ in the definition of $E^{(1)}$. Denote $(E^{(1)})^{(1)}$ by $E^{(2)}$.

For $\alpha \in E^{(2)} \cup \{\kappa\}$ let $P^{(1)}[E \cap \alpha]$ be a set of all pairs $\langle C_0, C_1\rangle$ so that C_0, C_1 are closed subsets of E, $C_0 \supseteq C_1$ and for every $\beta \in C_1$, $C_0 \cap \beta$ is a $V[G_\beta]$-generic subset of $P[E \cap \beta]$.

Lemma 4.2. *For every $\alpha \in (E^{(1)})_0$, $P^{(1)}[E \cap \alpha]$ is an (α,∞)-distributive forcing notion in $V[G_\alpha]$.*

Proof. Let $\beta < \alpha$ and $\langle D_\gamma \mid \gamma < \beta\rangle$ be a sequence of dense open subsets of $P^{(1)}[E \cap \alpha]$ in $V[G_\alpha]$. There exists a closed unbounded subset $C \in V$ of α so that for every $\nu \in C$

$$N_\nu = \langle V_\nu[G_\nu], \in, \mathscr{P}_\nu, E \cap \nu, \langle \boldsymbol{D}_\gamma \mid \gamma < \nu\rangle\rangle \prec \langle V_\alpha[G_\alpha], \in, \mathscr{P}_\alpha, E \cap \alpha, \langle \boldsymbol{D}_\gamma \mid \gamma < \nu\rangle\rangle.$$

Since $\mathscr{P}_\alpha$ satisfies α-c.c. and α is a measurable in V, $\alpha \in (E^{(1)})_0$, hence $(E^{(1)})(\beta) \cap \alpha$ is stationary.

Pick $\gamma \in (E^{(1)})(\beta) \cap \{\xi \mid \xi \text{ is a limit point of } C\}$. Then b_γ (the generic sequence to γ) has order type $\geqq \beta$, and starting from some place $\delta < \gamma$ is contained in the club $C \cap \gamma$ intersected with $E^{(1)} \cap \gamma$, which is a stationary subset of γ.

Let $\langle \beta_\xi \mid \xi < \beta\rangle$ be the increasing continuous enumeration of the first β members of $b_\gamma - \delta$.

Define in $V[G_{\gamma+1}]$ a sequence $\langle q_\nu \mid \nu < \beta\rangle$ so that

(1) $q_\nu \in P^{(1)}[E \cap \beta_{\nu+1}] \cap V[G_{\beta_\nu+1}]$,

(2) $q_\nu = \langle C_0, C_1\rangle$ and $\max C_{i\nu} = \beta_\nu$ for $i = 0, 1$,

(3) $q_{\nu+1} \in D_\nu$,

(4) $q_{\nu+1} \geqq q_\nu$.

Consider first the successor stage. Inside $N_{\beta_{\nu+1}}$ find some $q'_\nu \geqq q_\nu$, $q'_\nu \in D_\nu$. Since $\beta_{\nu+1} \in E^{(1)}$ by Lemma 4.1 there exists $q_{\nu+1} \geqq q'_\nu$ satisfying (2).

Let now ν be a limit ordinal. Set $q_\nu = \langle C_{0\nu}, C_{1\nu}\rangle$ where $C_{i\nu} = \bigcup_{\nu'<\nu} C_{i\nu'} \cup \{\beta_\nu\}$ for $i = 0, 1$. As in Lemma 4.1, such defined $C_{0\nu}$ will be a $V[G_{\beta_\nu}]$-generic club of $P[E \cap \beta_\nu]$. Hence $q_\nu \in P^{(1)}[E \cap \beta_{\nu+1}]$.

It completes the definition of $\langle q_\nu \mid \nu < \beta\rangle$.

Set $q = \langle C_0, C_1\rangle$, where $C_i = \bigcup_{\nu\in\beta} C_{i\nu} \cup \{\bigcup\{\beta_\xi \mid \xi < \beta\}\}$ for $i = 0, 1$. Then $q \in \bigcap_{\nu<\beta} D_\nu$. □

LEMMA 4.3. *For every* $\alpha \in E^{(2)}$ *there exists a* $V[G_\alpha]$*-generic subset of* $P^{(1)}[E \cap \alpha]$ *in* $V[G_{\alpha+1}]$.

The proof is as that of Lemma 4.1; just use Lemma 4.2 for nonlimit stages.

It is possible to prove even more.

LEMMA 4.4. *Let* $\alpha \in E^{(2)}$ *and let* $C_0 \in V[G_{\alpha+1}]$ *be a* $V[G_\alpha]$*-generic club of* $E \cap \alpha$ *defined as in Lemma* 4.1. *Then there exists a club* $C_1 \in V[G_{\alpha+1}]$ *so that the set* $\{\langle c_0, c_1\rangle \mid c_i$ *is an initial segment of* C_i *for* $i = 0, 1\}$ *is a* $V[G_\alpha]$*-generic subset of* $P^{(1)}[E \cap \alpha]$.

PROOF. Suppose first $\alpha \in (E^{(2)})_0$. Then in $V[G_{\alpha+1}]$, $(\alpha^+)^V = \bigcup_{i\in\omega} B_i$ where each $B_i \in V[G_\alpha]$ and $|B_i|^{V[G_\alpha]} < \alpha$.

Clearly, $P^{(1)}[E \cap \alpha] \simeq P[E \cap \alpha] * Q$ for some (α, ∞)-distributive forcing notion Q. The forcing with $P[E \cap \alpha]$ preserves cardinals. So we can represent the set of all dense subsets of Q of $V[G_\alpha, C_0^l]$ as a union of countably many sets each belonging to $V[G_\alpha, C_0]$ and of cardinality less than α. Hence, it is possible to construct a $V[G_\alpha, C_0]$-generic subset of Q in $V[G_{\alpha+1}]$. Just use the sequence $\langle B_n : n < \omega\rangle$ and that $(\alpha^+)^V = (\alpha^+)^{V[G_\alpha, C_0]}$. Hence there exists a club $C_1 \in V[G_{\alpha+1}]$ that satisfies the claim of the lemma.

Let now $\alpha \in E^{(2)} - (E^{(2)})_0$. Then as in Lemma 4.1, generic using the generic sequence b_α, we construct the union of generic clubs that will satisfy the claim of the lemma. □

Using Lemmas 4.1–4.4, the continuation of the construction became the routine translation of [G 2] to our case. We leave the details to the reader.

To construct the model with NS_{μ^+} precipitous, for some regular μ, we do not need really a coherent sequence on κ of the length κ. A coherent sequence of length $\mu + 1$ is enough. Just after the forcing with $\mathcal{P}_\kappa$ collapse κ to μ^+ using the Levy collapse. Every $E \in \bigcap_{\alpha<\mu} U(\kappa, \alpha)$ will then be a fat subset of μ^+. Now apply the arguments of Lemmas 4.1–4.4 and [G 2].

§5. $NS_\kappa \restriction S$ saturated for S containing all the cofinalities

Assume G.C.H. Let $\vec{U}$ be a coherent sequence of the length $l^{\vec{U}} = \kappa + 1$ so that $0^{\vec{U}}(\kappa) = \kappa$ and $U(\kappa, 0)$ concentrates on α^+-supercompact cardinals.

As in Section 4, let

$$A = \{\alpha < \kappa \mid \alpha \in \mathrm{dom}_1 \vec{U}, \alpha \text{ is an } \alpha^+\text{-supercompact cardinal}\}.$$

Then $A \in U(\kappa, 0)$ and w.l.o.g. for every $\alpha \in \mathrm{dom}_1 \vec{U}$, $A \cap \alpha \in U(\alpha, 0)$. For $\alpha \in A$ fix a normal measure $U(\alpha)$ over $\mathcal{P}_\alpha(\alpha^+)$.

Let us define now the preparation forcing notion $\mathscr{P}_\kappa$. Basically $\mathscr{P}_\kappa$ will be as in Section 4 but, in addition, we shall add α^+-Cohen subsets to α, for every inaccessible α below κ.

Define by induction the interation $\mathscr{P}_\alpha$ for $\alpha \in$ the closure of $\{\beta < \kappa \mid \beta$ is an inaccessible or $\beta = \gamma + 1$ and γ is an inaccessible$\}$. Let us concentrate on successor stages. For limit stages we refer to §§1–3. So, suppose $\mathscr{P}_\alpha$ is defined and α is an inaccessible. Define $\mathscr{P}_{\alpha+1}$.

Case 1. $\alpha \notin A \cup \mathrm{dom}_1 \vec{U}$

Let $C(\alpha^+) = \{f \in V^{\mathscr{P}_\alpha} \mid f$ is a partial function from $\alpha^+ \times \alpha$ into $\{0,1\}$, $|f|^{V^{\mathscr{P}_\alpha}} < \alpha\}$, i.e., the forcing for adding α^+-Cohen subsets to α.

Set $\mathscr{P}_{\alpha+1} = \mathscr{P}_\alpha * C(\alpha^+)$.

Case 2. $\alpha \in A$

In this case we shall first add α^+-Cohen subsets of α, then define a fine ultrafilter over $\mathscr{P}_\alpha(\alpha^+)$ and use it to change cofinalities of α, α^+ to ω. Let $\langle \boldsymbol{A}_\beta \mid \beta < \alpha^{++} \rangle$ be the W-least enumeration of all canonical $\mathscr{P}_\alpha * C(\alpha^+)$-names of subsets of $\mathscr{P}^V_\alpha(\alpha^+)$. Let

$$j\colon V \to N \simeq V^{\mathscr{P}_\alpha(\alpha^+)}/U(\alpha)$$

be the canonical elementary embedding. Let G_α be a generic subset of $\mathscr{P}_\alpha$ and G be a $V[G_\alpha]$-generic subset of $C(\alpha^+)$. In $N[G_\alpha * G]$ pick some $q \in C((j(\alpha^+))$, $q \supseteq j''(G)$.

$$j(\mathscr{P}_\alpha * C(\alpha^+)) = \mathscr{P} * C(\alpha^+) * Q * Q',$$

where $C(\alpha^+) * Q$ is the forcing used on κ in N and $Q' = \mathscr{P}_{j(\alpha)} * C(j(\alpha^+))/\mathscr{P}_{\alpha+1}$. Now using Lemma 1.2, define in $V[G_\alpha * G]^Q$ an E-increasing sequence $\langle p_\beta \mid \beta < \alpha^{++} \rangle$ of E-extensions of q so that in $N[G_\alpha * G]^Q$

$$p_{\beta+1} \Vdash_{Q'} j''(\alpha^+) \in j(\boldsymbol{A}_\beta) \qquad \text{for every } \beta < \alpha^{++}.$$

Define now a fine ultrafilter $\bar{U}(\alpha)$ extending $U(\alpha)$ in $V[G_\alpha * G]$ as follows. Set $C \in \bar{U}(\alpha)$, if $C \subseteq \mathscr{P}_\alpha(\alpha^+)$ and for some $\beta < \alpha^{++}$, $r \in G_\alpha * G$, in N

$$r \Vdash_{\mathscr{P}_\alpha * C(\alpha^+)} (\boldsymbol{p}_\beta \Vdash_{Q'} j''(\check{\alpha}^+) \in j(\boldsymbol{C}))$$

in case $Q = \varnothing$, or if $Q \neq \varnothing$, then for some T s.t. $\langle \varnothing, T \rangle \in Q$

$$r^\frown \langle \varnothing, T \rangle \Vdash_{\mathscr{P}_{\alpha+1}} (\boldsymbol{p}_\beta \Vdash_{Q'} j''(\check{\alpha}^+) \in j(\boldsymbol{C})).$$

The checking that such defined $\bar{U}(\alpha)$ is a fine ultrafilter is as in Lemmas 1.5 and 2.2.

Let, as in §4, Q_α be the strongly compact Prikry forcing with $\bar{U}(\alpha)$ in $V[G_\alpha * G]$. Define $\mathcal{P}_{\alpha+1}$ to be $\mathcal{P}_\alpha * (C(\alpha^+) * Q_\alpha)$. Notice that the forcing $C(\alpha^+) * Q_\alpha$ is isomorphic to the following forcing notion, $Q = \{\langle t, q, \boldsymbol{T}\rangle \mid t \in V$ is a finite sequence, $q \in C(\alpha^+)$, $q \Vdash \langle \check{t}, \boldsymbol{T}\rangle \in Q_\alpha\}$. Q is clearly α-weakly closed and has the Prikry property.

Case 3. $\alpha \in \operatorname{dom}_1 \vec{U}$

Force first with $C(\alpha^+)$ and then define $\mathcal{P}_{\alpha+1}$ as in §3. The additional arguments needed here are really contained in case 2. $\mathcal{P}_{\alpha+1}$ will be $\mathcal{P}_\alpha * (C(\alpha^+) * \mathcal{P}(\alpha, 0^{\vec{U}}(\alpha))$. As in case 2, $C(\alpha^+) * \mathcal{P}(\alpha, 0^{\vec{U}}(\alpha))$ is isomorphic to α-weakly closed forcing satisfying the Prikry condition.

Let

$$S = \{\alpha \in \operatorname{dom}_1 \vec{U} \mid \text{for some } \beta, 0 < \beta < \kappa, 0^{\vec{U}}(\alpha) = \beta + \beta\}.$$

Then $S \in \bigcap_{0<\beta<\kappa} U(\kappa, \beta + \beta)$. Denote by S_β the set $\{\alpha \in S \mid 0^{\vec{U}}(\alpha) = \beta + \beta\}$, for $0 < \beta < \kappa$. Clearly every S_β is a stationary subset of κ. Let G_κ be a generic subset of $\mathcal{P}_\kappa$. Then every S_β remains stationary in $V[G_\kappa]$, since $\mathcal{P}_\kappa$ satisfies κ-c.c. Also, for every regular cardinal $\mu < \kappa$ of $V[G]$, every $\alpha \in S_\mu$ has cofinality μ in $V[G]$.

We shall define a generic extension of $V[G_\kappa]$ in which the filter $\bigcap_{0<\beta<\kappa} U(\kappa, \beta + \beta)$ extends to the closed unbounded filter restricted to S. Notice that $\kappa - S$ is a fat subset of κ, in $V^{\mathcal{P}_\kappa}$. Since for every regular $\mu < \kappa$, $\alpha < \kappa$ with $0^{\vec{U}}(\alpha) = \mu$ a closed unbounded subset of the generic sequence b_α is disjoint from S. So, by [A–S], the forcing notion $P[\kappa - S]$ (see §4 for the definition) is (κ, ∞)-distributive. We shall use Woodin's trick first to destroy a stationary set and then to shoot clubs avoiding its subsets by closed forcing. The forcing $P[\kappa - S] * C(\kappa^+)$ will destroy S and add clubs avoiding subsets of S. We shall embed $P[\kappa - S] * C(\kappa^+)$ into the forcing notion $C(\kappa^+) * \mathcal{P}(\kappa, \alpha)$ for every $\alpha < \kappa$. Namely, we shall show how to pick in $V^{\mathcal{P}_\kappa * C(\kappa^+) * \mathcal{P}(\kappa,\alpha)}$ a $V^{\mathcal{P}_\kappa * C(\kappa^+)}$-generic subset of $P[\kappa - S]$. Notice that

$$P[\kappa - S] * C(\kappa^+) = C(\kappa^+) \times P[\kappa - S].$$

Set $S' = \{\alpha < \kappa \mid \text{in } V, \alpha \text{ is a measurable and for every } \gamma < \alpha\ \{\beta < \alpha \mid 0^{\vec{U}}(\beta) \geqq \gamma\}$ is a stationary subset of $\alpha\}$. Then $S' \in \bigcap_{\beta<\kappa} U(\kappa, \beta)$ and for every $\alpha \in S' \cup \{\kappa\}$, $\alpha - S$ is a fat subset of α in $V[G_\alpha]$, where $G_\alpha = G_\kappa \cap \mathcal{P}_\alpha$ for some fixed generic subset G_κ of $\mathcal{P}_\kappa$. Define by induction the following sets:

$$S(0) = \{\alpha \in S' \mid \alpha \in A\} \qquad \text{for } \beta, \quad 0 < \beta \leqq \kappa;$$

$$S(\beta) = \{\alpha \in S' \mid 0^{\vec{U}}(\alpha) = \beta \text{ and for every } \delta < \beta, S(\delta) \cap \alpha \in U(\alpha, \delta)\}.$$

Set $S^* = \bigcup_{\beta<\kappa} S(\beta)$. Then $S^* \in \bigcap_{\beta<\kappa} U(\kappa,\beta)$. Let $G(\alpha^+) = G_\kappa \cap C(\alpha^+)$ for every $\alpha \in S^*$.

LEMMA 5.1. *For every $\alpha \in S^*$ there exists a $V[G_\alpha * G(\alpha^+)]$-generic subset of $P[\alpha - S]$ in $V[G_{\alpha+1}]$.*

PROOF. Note first that for $\alpha \in S^*$, $\alpha - S$ is a fat subset of α in $V[G_\alpha * G(\alpha^+)]$, since $\alpha - S$ is fat in $V[G_\alpha]$ and the forcing $C(\alpha^+)$ is α-closed. By [A–S] the forcing $P[\alpha - S]$ is an (α,∞)-distributive forcing notion in $V[G_\alpha * G(\alpha^+)]$.

If $\alpha \in S(0)$, then the strongly compact Prikry forcing was used on α and so $\operatorname{cof}\alpha = \operatorname{cof}\alpha^+ = \omega$ in $V[G_{\alpha+1}]$. There is no new bounded subsets of α in $V[G_{\alpha+1}]$. Then, as in Lemma 4.1, there exists a $V[G_\alpha * G(\alpha^+)]$-generic subset of $P[\alpha - S]$ in $V[G_{\alpha+1}]$.

Suppose now that for every $\gamma < \beta < \kappa$, $\alpha \in E_\gamma$ the lemma is proved. Let us prove it for β. Consider the generic sequence b_α to α. Clearly, $d_\alpha = b_\alpha - S$ will be closed unbounded in α. Pick some $\xi < \alpha$ so that $d_\alpha - \xi$ is contained in every $S(\delta) \cap \alpha$ $(\delta < \beta)$. Denote $d_\alpha - \xi$ by d. Let $\langle \gamma_\xi \mid \xi < \tau \rangle$ be the increasing continuous enumeration of d.

Define by induction an increasing continuous sequence of closed sets $\langle c_\xi \mid \xi < \tau \rangle$ so that

(1) $c_\xi \in V[G_{\gamma_\xi+1}]$ is a $V[G_{\gamma_\xi} * G(\gamma_\xi^+)]$-generic club through $\gamma_\xi - S$,

(2) $c_{\xi+1}$ is an endextension of c_ξ,

(3) for a limit ξ, $c_\xi = \bigcup\{c_{\xi'} \mid \xi' < \xi\}$.

We define $c_{\xi+1}$ using the inductive assumption. Let us show that for a limit ξ, $c_\xi = \bigcup\{c_{\xi'} \mid \xi' < \xi\}$ is $V[G_{\gamma_\xi} * G(\gamma_\xi^+)]$-generic. Let $D \in V[G_{\gamma_\xi} * G(\gamma_\xi^+)]$ be a dense open subset of $P[\gamma_\xi - S]$. Then for some $\eta < (\gamma_\xi^+)^{V[G_{\gamma_\xi}]} = (\gamma_\xi^+)^V$, $D \in V[G_{\gamma_\xi} * G(\gamma_\xi^+) \restriction \eta]$, where $G(\gamma_\xi^+) \restriction \eta$ are η-first Cohen generic subsets of γ_ξ, since $P[\gamma_\xi - S]$ is a set of cardinality γ_ξ. Denote by $C(\eta)$ the forcing for adding η-Cohen subsets to γ_ξ. Let us assume for simplification of the notations that $\eta = \gamma_\xi$. Otherwise just use an appropriate coding of η by a subset of γ_ξ. By the choice of ultrafilters $U(\gamma_\xi, \nu, t)$ $(\nu < 0^{\bar{U}}(\gamma_\xi)$, t a ν-coherent sequence)

$$\{\alpha < \gamma_\xi \mid (G(\gamma_\xi) \restriction \alpha) \cap \alpha = G(\alpha^+) \restriction \alpha\} \in U(\gamma_\xi, \nu, t).$$

Then starting with some $\xi' < \xi$, d will be contained in this set. For the same reason, for some ξ^*, $\xi' \leqq \xi^* < \xi$, for every ξ'', $\xi^* \leqq \xi'' < \xi$,

$$D \cap P[\gamma_{\xi''} - S] \in V[G_{\gamma_{\xi''}} * G(\gamma_{\xi''}^+)]$$

and it is a dense subset of $P[\gamma_{\xi''} - S]$. Then $c_{\gamma_{\xi''}}$ extends some element of

$D \cap P[\gamma_{\xi^*} - S]$. So, $c_{\gamma_{\xi^*}} \cup \{\gamma_{\xi^*}\}$ belongs to D. Hence, also c_{γ_ξ} is an extension of an element of D. So c_{γ_ξ} is a $V[G_{\gamma_\xi}]$-generic club. It completes the induction. □

For $\gamma < \kappa$, denote by j^γ the canonical elementary embedding of V into $N^\gamma \simeq V^\kappa / U(\kappa, \gamma)$.

Applying Lemma 5.1 to κ in N^γ we obtain the following.

LEMMA 5.2. *Let $\gamma < \kappa$, $G_{\kappa+1}$ be a generic subset of $\mathcal{P}_{\kappa+1}$ over N^γ. Then in $N^\gamma[G_{\kappa+1}]$ there exists a $V[G_\kappa * G(\kappa^+)]$-generic subset of $P[\kappa - S]$.*

For a forcing notion P let us denote by RO(P) the complete Boolean algebra of regular open subsets of P.

LEMMA 5.3. *For every $\gamma < \kappa$, $\mathrm{RO}(P[\kappa - A] * C(\kappa^+))$ is isomorphic with a complete subalgebra of $\mathrm{RO}(C(\kappa^+) * \mathcal{P}(\kappa, \gamma))$.*

PROOF. In N^γ the forcing used on κ is $C(\kappa^+) * \mathcal{P}(\kappa, \gamma)$. Let $G(\kappa^+) * G'$ be a $V[G_\kappa]$-generic subset of $C(\kappa^+) * \mathcal{P}(\kappa, \gamma)$. By Lemma 5.2, in $V[G_\kappa * G(\kappa^+) * G']$ there exists a $V[G_\kappa * G(\kappa^+)]$-generic subset G'' of $P[\kappa - S]$. Then $G'' * G(\kappa^+)$ will be a $V[G_\kappa]$-generic subset of $P[\kappa - S] * C(\kappa^*)$. □

Further, we shall identify $P[\kappa - S] * C(\kappa^+)$ with subalgebras of $C(\kappa^+) * \mathcal{P}(\kappa, \alpha)$ isomorphic to it.

We are now ready to define the main forcing. It will be a subordering of $P[\kappa - S] * C(\kappa^*)$.

Work in $V[G_\kappa]$. Let X be the set of all $P[\kappa - S] * C(\kappa^+)$-names $\boldsymbol{a}$ of pairs $\langle p, \alpha\rangle$ so that $\alpha < \kappa$, $p \in P[\kappa - S] * C(\kappa^+)$ and if $\langle p_1, \alpha\rangle, \langle p_2, \alpha\rangle \in \boldsymbol{a}$, $p_1 \neq p_2$ then p_1, p_2 are incompatible. Clearly, every subset of κ in $V[G_\kappa]^{P[\kappa - S] * C(\kappa^+)}$ has a name in X. Since $P[\kappa - S] * C(\kappa^+)$ satisfies κ^+-c.c., the cardinality of X is κ^+. Let $\langle \boldsymbol{a}_\alpha \mid \alpha < \kappa^+\rangle$ be an enumeration of X such that every $\boldsymbol{a} \in X$ appears κ^+-many times. Let also $\boldsymbol{a}_0 = \check{S}$.

Define by induction sequences $\langle B_\alpha \mid \alpha < \kappa^+\rangle$, $\langle \boldsymbol{p}^\gamma_\delta \mid \delta < \kappa^+, \gamma = \gamma' + \gamma'$ for some γ', $0 < \gamma' < \kappa\rangle$, $\langle \boldsymbol{q}^\gamma_\delta \mid \delta < \kappa^+, \gamma = \gamma' + \gamma'$ for some γ', $0 < \gamma' < \kappa\rangle$ so that

(1) $\langle B_\alpha \mid \alpha < \kappa^+\rangle$ is an increasing sequence of complete subalgebras of $P[\kappa - S] * C(\kappa^+)$,

(2) for every γ s.t. $\gamma = \gamma' + \gamma'$ for some γ', $0 < \gamma' < \kappa$, $\langle p^\gamma_\delta \mid \delta < \kappa^+\rangle$ is an E-increasing sequence of elements of

$$j^\gamma(\mathcal{P}_\kappa * C(\kappa^+) * \mathcal{P}(\kappa, \gamma)) / \mathcal{P}_\kappa * C(\kappa^+) * \mathcal{P}(\kappa, \gamma),$$

(3) for every γ s.t. $\gamma = \gamma' + \gamma'$, for some γ', $0 < \gamma' < \kappa$, for every $\delta < \kappa^+$, $\boldsymbol{q}^\gamma_\delta$ is a $p[j^\gamma(\kappa - S)]$-name of an element of B_δ. For $\delta_1 < \delta_2 < \kappa^+$, $\boldsymbol{q}^\gamma_{\delta_2}$ is forced to be stronger than $\boldsymbol{q}^\gamma_{\delta_1}$.

Set $B_0 \neq \emptyset$, $p_0^\gamma = \emptyset$, $q_0^\gamma = \emptyset$. Let $B_1 = \mathrm{RO}(P[(\kappa - S) \cup S])$, $p_1^\gamma = \emptyset$. Define q_1^γ. B_1 is naturally embeddable into $\mathrm{RO}(P[\kappa - S] * C(\kappa^+))$. We identified $\mathrm{RO}(P[\kappa - S] * C(\kappa^+))$ with the complete subalgebra of $\mathrm{RO}(C(\kappa^+) * \mathscr{P}(\kappa, \gamma))$. So in $N^\gamma[G_\kappa]^S, j^\gamma(\mathscr{P}_\kappa)/G_\kappa$ we have the canonical name $\boldsymbol{G}(B_1)$ of a generic subset of B_1. Set

$$q_1^\gamma = \bigcup \boldsymbol{G}(B_1) \cup \{\kappa\} \cup \{\min \boldsymbol{G}(P[j^\gamma(\kappa - S)]) - \kappa\},$$

where $G(P[j^\gamma(\kappa - S)])$ is the canonical $j^\gamma(\mathscr{P}_\kappa)$-name of a generic subset of $P[j^\gamma(\kappa - S)]$.

Before turning to the general case, let us do one more step and define B_2, p_2^γ, q_2^γ. Consider a_1. If a_1 is not a B_1-name, then set $B_2 = B_1$, $p_2^\gamma = p_1^\gamma$ and $q_2^\gamma = q_1^\gamma$. Suppose now that a_1 is a B_1-name. For every $\gamma = \gamma' + \gamma'$, $0 < \gamma' < \kappa$, let $\bar{q}^\gamma$ be a $P[j^\gamma(\kappa - S)]$-name of an element of $j^\gamma(B_1)$ above q_1^γ deciding the statement "$\kappa \in j^\gamma(a_1)$". Let also $\bigcup \bar{q}^\gamma \in \boldsymbol{G}(P[j^\gamma(\kappa - S)])$. Pick p_2^γ to be an E-extension of p_1 forcing "$\bar{q}^\gamma \Vdash (\kappa \in j^\gamma(a_1))$" or "$\bar{q}^\gamma \Vdash (\kappa \notin j^\gamma(a_1))$". It exists by Lemma 1.4. Notice that we identify $P[j^\gamma(\kappa - S)]$ with a complete subalgebra of $C(j^\gamma(\kappa^+)) * \mathscr{P}(j^\gamma(\kappa), \gamma)$. It does not matter here which embedding we pick. Moreover, $C(j^\gamma(\kappa^+)) * \mathscr{P}(j^\gamma(\kappa), \gamma)$ can be replaced by any other forcing including $P[j^\gamma(\kappa - S)]$ and satisfying the Prikry condition.

If for some $\gamma = \gamma' + \gamma'$, $0 < \gamma' < \kappa$,

$$\| p_2^\gamma \Vdash (\bar{q}^\gamma \Vdash \kappa \in j^\gamma(a_1)) \|^{C(\kappa^+) * \mathscr{P}(\kappa, \gamma)/B_1} \neq 1$$

then set $B_2 = B_1$, $q_2^\gamma = \bar{q}^\gamma$.

Suppose otherwise, i.e. for every $\gamma = \gamma' + \gamma'$, $0 < \gamma' < \kappa$,

$$1 = \| p_2^\gamma \Vdash (\bar{q}^\gamma \Vdash \kappa \in j^\gamma(a_1)) \|^{C(\kappa^+) * \mathscr{P}(\kappa, \gamma)/B_1}.$$

Then set $B_2 = B_1 * \mathrm{RO}(P[(\kappa - S) \cup a_1])$. B_2 is embeddable into $\mathrm{RO}[P(\kappa - S] * C(\kappa^+))$ by an embedding extending the embedding of B_1, since after the forcing with $P[\kappa - S]$, B_1, B_2 became κ-closed forcing notions of cardinality κ and hence they are isomorphic to the Cohen generic subsets of κ. Set

$$q_2^\gamma = \langle \bar{q}^\gamma, \bigcup \boldsymbol{G}(P[(\kappa - S) \cup a_1]) \cup \{\kappa\} \cup \{\min \boldsymbol{G}(P[j^\gamma(\kappa - S)]) - \kappa\} \rangle,$$

where $\boldsymbol{G}(P[(\kappa - S) \cup a_1])$ is the canonical name of a generic subset of $P[(\kappa - S) \cup a_1]$.

Let us give now the definitions in general case. Let $\alpha < \kappa^+$ and below α everything is defined. For successor ordinal α the definitions are as in the case $\alpha = 2$. Suppose now that α is a limit ordinal. Set B_α to be the direct limit of

$\langle B_\beta \mid \beta < \alpha\rangle$ if $\operatorname{cof}\alpha = \kappa$, or B_α = the inverse limit of $\langle B_\beta \mid \beta < \alpha\rangle$ if $\operatorname{cof}\alpha < \kappa$. Notice that $C(\kappa^+)$ is isomorphic to the iteration of Cohen subsets of κ with $< \kappa$-support, i.e. the direct limit, for $\operatorname{cof}\alpha = \kappa$, and the inverse limit for $\operatorname{cof}\alpha < \kappa$. So B_κ $(\alpha < \kappa^+)$ will be embeddable into $\mathrm{RO}(P[\kappa - S] * C(\kappa^+))$ by an embedding extending the embeddings of B_β's for $\beta < \alpha$. Let $\gamma = \gamma' + \gamma'$ for some γ', $0 < \gamma' < \kappa$. Set $\boldsymbol{P}^\gamma_\alpha$ a name of the E-extension of $\langle \boldsymbol{p}^\gamma_\beta \mid \beta < \alpha\rangle$. It exists by Lemma 1.2. Finally, set $\boldsymbol{q}^\gamma_\alpha$ = the coordinate union of $\langle \boldsymbol{q}^\gamma_\beta \mid \beta < \alpha\rangle$. Then $\boldsymbol{q}^\gamma_\alpha$ will be a $P[j^\gamma(\kappa - S)]$-name of an element of B_α, since for every $\beta < \alpha$, $\max \boldsymbol{q}^\gamma_\beta \in G(P[j^\gamma(\kappa - S)])$ which is a closed unbounded subset of $P[j^\gamma(\kappa - S)]$ and $\boldsymbol{q}^\gamma_\beta$ is allowed to enter $j^\gamma(\kappa - S)$. It completes the definition.

Set B to be the direct limit of $\langle B_\alpha \mid \alpha < \kappa^+\rangle$. Let $G(B)$ be the $V[G_\kappa]$-generic subset of B. Let $U^* = \{x \in \mathcal{P}(\kappa)^{V[G_\kappa * G(B)]} \mid$ for some $\alpha < \kappa^+$, for some $\mathcal{P}_\kappa * B$-name $\boldsymbol{x}$ of x, for every $\gamma = \gamma' + \gamma'$, $0 < \gamma' < \kappa$,

$$1 = \|\boldsymbol{p}^\gamma_\alpha \Vdash (\boldsymbol{q}^\gamma_\alpha \Vdash \check{\kappa} \in j^\gamma(\boldsymbol{x}))\|^{C(\kappa^+) * \mathcal{P}(\kappa,\gamma)/G(B)}\}.$$

Lemma 5.4. *U^* is the closed unbounded filter over κ restricted to S.*

Proof. Let us show first that U^* contains the closed unbounded filter over κ restricted to S. Let $x \subseteq \kappa$ be such that for some club $C \subseteq \kappa$, $x \cap S \supseteq S \cap C$. There exists $\alpha < \kappa^+$ so that $x, C \in V[G_\kappa * G(B_\alpha)]$, where $G(B_\alpha) = G(B) \cap B_\alpha$, since B is a complete subalgebra of $\mathrm{RO}(C(\kappa^+) * P[\kappa - S])$ satisfying κ^+-c.c. But then

$$1 = \|\boldsymbol{p}^\gamma_\alpha \Vdash (\boldsymbol{q}^\gamma_\alpha \Vdash \check{\kappa} \in j^\gamma(S \cap C) \supseteq j^\gamma(\check{S} \cap \boldsymbol{x}))\|^{C(\kappa^+) * \mathcal{P}(\kappa,\gamma)/G(B_\alpha)}.$$

Hence $x \in U^*$.

Suppose now that $x \in U^*$. Then for some $\alpha < \kappa^+$, $x \in V[G_\kappa * G(B_\alpha)]$. Let $\beta \geqq \alpha$ be such that

$$1 = \|\boldsymbol{p}^\gamma_\beta \Vdash (\boldsymbol{q}^\gamma_\beta \Vdash \check{\kappa} \in j^\gamma(\boldsymbol{x}))\|^{C(\kappa^+) * \mathcal{P}(\kappa,\gamma)/G(B_\beta)}.$$

Then for some $\delta \geqq \beta$, $B_{\delta+1} = B_\delta * P[(\kappa - S) \cup \boldsymbol{x}]$. Hence $S \cap x$ contains a club intersected with S in $V[G_\kappa * G(B_\delta)]$. □

Lemma 5.5. *U^* is the κ^+-saturated filter in $V[G_\kappa * G(B)]$.*

Proof. The set $S \in U^*$ and S is the disjoint union of S_γ's, where $\gamma = \gamma' + \gamma'$, $0 < \gamma' < \kappa$, $S_\gamma = \{\alpha < \kappa \mid 0^{\bar{U}}(\alpha) = \gamma\}$. It is enough to show that for every $\gamma = \gamma' + \gamma'$, $0 < \gamma' < \kappa$,

$$U^* \restriction S_\gamma = \{x \subseteq \kappa \mid (S_\gamma \cap x) \cup (\kappa - S_\gamma) \in U^*\}$$

is κ^+-saturated.

Let us fix γ. Suppose $x \subseteq \kappa$ is a $U^* \restriction S_\gamma$-positive set. Pick some $\alpha < \kappa^+$ so that $x \in V[G_\kappa * G(B_\alpha)]$ and $\boldsymbol{a}_\alpha$ is a B_α-name of $(S_\gamma - x) \cup (\kappa - S)$. Then for every $\delta \neq \gamma$

$$1 = \| \boldsymbol{p}^{\delta}_{\alpha+1} \Vdash (\boldsymbol{q}^{\delta}_{\alpha+1} \Vdash \check{\kappa} \in j^{\delta}(\boldsymbol{a}_\alpha)) \|^{C(\kappa^+) * \mathcal{P}(\kappa,\gamma)/G(B_\alpha)}.$$

Since $S_\gamma - x \notin U^* \restriction S_\gamma$,

$$1 \neq \| \boldsymbol{p}^{\gamma}_{\alpha+1} \Vdash (\boldsymbol{q}^{\gamma}_{\alpha+1} \Vdash \check{\kappa} \in j^{\delta}(\boldsymbol{a}_\alpha)) \|^{C(\kappa^+) * \mathcal{P}(\kappa,\gamma)/G(B_\alpha)}.$$

Hence for every $U^* \restriction S_\gamma$-positive set x there exists a condition $t_x \in C(\kappa^+) * \mathcal{P}(\kappa, \gamma)/G(B)$ and $\alpha < \kappa^+$ so that

$$t_x = \| \boldsymbol{p}^{\gamma}_{\alpha} \Vdash (\boldsymbol{q}^{\gamma}_{\alpha} \Vdash \check{\kappa} \in j^{\gamma}(\boldsymbol{x})) \|^{C(\kappa^+) * \mathcal{P}(\kappa,\gamma)/G(B)}.$$

If x_1, x_2 are $U^* \restriction S_\gamma$-positive and $x_1 \cap x_2$ is not $U^* \restriction S$-positive then t_{x_1}, t_{x_2} are incompatible. The forcing $C(\kappa^+) * \mathcal{P}(\kappa, \gamma)/G(B)$ satisfies κ^+-c.c. Hence $U^* \restriction S$ is κ^+-saturated. □

Lemma 5.6. *For every $\gamma = \gamma' + \gamma'$, $0 < \gamma' < \kappa$, S_γ remains stationary in $V[G * G(B)]$.*

Proof. Suppose that for some γ there exists $C \subseteq \kappa$ club $C \cap S_\gamma = 0$. Pick an $\alpha < \kappa^+$ so that $C \in V[G_\kappa * G(B_\alpha)]$. Then, in $N^\gamma[G_\kappa * G(B_\alpha)]$

$$\boldsymbol{q}^{\gamma}_{\alpha} \Vdash (\check{\kappa} \in j^{\gamma}(\boldsymbol{C}) \cap j^{\gamma}(S_\gamma) \text{ and } j^{\gamma}(\boldsymbol{C}) \cap j^{\gamma}(S_\gamma) = \varnothing)$$

which is impossible. □

Theorem 5.7. *In $V[G_\kappa * G(B_\kappa)]$, $\mathrm{NS}_\kappa \restriction S$ is κ^+-saturated and for every regular $\alpha < \kappa$, $S \cap \{\beta < \kappa \mid \operatorname{cf} \beta = \alpha\}$ is stationary.*

Added in proof.

1. S. Shelah [S] defined a generalization of the iteration of §1 which it is possible to use for small cardinals.

2. It is impossible completely to remove the assumptions on U_0 in Theorem I, but it is possible to replace them by some which do not even require $\exists \kappa$ $o(\kappa) = \kappa^{++}$.

3. It is possible to drop the assumption that U_0 is concentrated on α^+-supercompact cardinals α in Theorem II. The idea is to start with a coherent sequence $\langle U_\alpha \mid \alpha < \kappa \rangle$ so that U_0 is a non-minimal Q-point and U_α ($\alpha > 0$) is normal. By [G 3] it is possible to obtain such from a measurable. Let $U_0' \leqq_{\mathrm{RK}} U_0$ be normal. Pick $A_0' \subseteq \kappa$ which belongs to $U_0 - U_0'$ and does not belong to

$\bigcup_{0<\alpha<\kappa} U_\alpha$. Let S be as in the proof of the theorem. For most α's in A'_0 force with $P[\alpha - S] * C(\alpha^+)$. For $\alpha \in \kappa - A'_0$ force as in §5. By the choice of A'_0, U_0 extends to a κ-complete ultrafilter U^*_0. U^*_0 will still be Q-point since the forcing satisfies κ-c.c. Then every Prikry sequence for U^*_0 is almost contained in any club of κ. So it is possible to pick a generic subset of $P[\alpha - S]$ as in §5. It is possible to iterate the forcing notions $P[\alpha - S]$ preserving the cardinals by the choice of S.

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